Mechanical Vibrations
Single DoF free vibration system

Single degree of freedom system with viscous damping:

1- Hysteresis Damping (Structural)
2- Viscously-Damped
3- Friction Damped
Single DoF free vibration system

Single degree of freedom system with viscous damping:

System

Free-body diagram
Single DoF free vibration system

Mathematical model (governing equation of motion):

\[ m \ddot{x} + c \dot{x} + kx = 0 \]

\[ c \dot{x} = F_d \] is the damping force.

It is proportional to velocity

Solution:
Assume \( x(t) = B e^{st} \) substitute into the equation of motion:

\[ ms^2 + cs + k = 0 \]

\[ s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]

\[ s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \]
Single DoF free vibration system

\[ x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t} \]

\[ x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \]

\[ = C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\} t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\} t} \]

Critical Damping Constant and the Damping Ratio.

\[ \left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \]

\[ c_c = 2m \sqrt{\frac{k}{m}} = 2 \sqrt{km} = 2m \omega_n \]
In the face of the value of the square root shown above three possible solutions are exist. These solutions are as follows:

\[ s_{1,2} = \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \omega_n \]
Solution # 1: under-damped vibration:
When $\zeta < 1$
The roots $S_{1,2}$ of the characteristic equation can now be written as:
$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega_n$$
The solution for this case becomes:
$$x(t) = C_1e^{(-\zeta+i\sqrt{1-\zeta^2})\omega nt} + C_2e^{(-\zeta-i\sqrt{1-\zeta^2})\omega nt}$$
$$= e^{-\zeta\omega nt} \left\{ C_1e^{i\sqrt{1-\zeta^2}\omega nt} + C_2e^{-i\sqrt{1-\zeta^2}\omega nt} \right\}$$
Real Initial Condition:

\[
= e^{-\xi \omega_n t} \left\{ (C_1 + C_2) \cos \sqrt{1 - \xi^2 \omega_n t} + i(C_1 - C_2) \sin \sqrt{1 - \xi^2 \omega_n t} \right\}
\]

\[
= e^{-\xi \omega_n t} \left\{ C_1' \cos \sqrt{1 - \xi^2 \omega_n t} + C_2' \sin \sqrt{1 - \xi^2 \omega_n t} \right\}
\]

\[
= X_0 e^{-\xi \omega_n t} \sin \left( \sqrt{1 - \xi^2 \omega_n t} + \phi_0 \right)
\]

Or:

\[
= X e^{-\xi \omega_n t} \cos \left( \sqrt{1 - \xi^2 \omega_n t} - \phi \right)
\]
Single DoF free vibration system

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$

\[ C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta \omega_n x_0}{\sqrt{1 - \zeta^2 \omega_n}} \]

\[
\begin{align*}
    x(t) &= e^{-\zeta \omega_n t} \left\{ x_0 \cos \sqrt{1 - \zeta^2 \omega_n} t \\
    &\quad + \frac{\dot{x}_0 + \zeta \omega_n x_0}{\sqrt{1 - \zeta^2 \omega_n}} \sin \sqrt{1 - \zeta^2 \omega_n} t \right\}
\end{align*}
\]
Single DoF free vibration system

Frequency of damped vibration:

\[ \omega_d = \sqrt{1 - \zeta^2} \omega_n \]
Solution # 2: critically damped vibration (ζ=1):

For this case the roots of the characteristic equation become:

\[ s_{1,2} = -\omega_n \]

Therefore the solution can be written as:

\[ x(t) = e^{-\omega_n t} \left\{ B_1 + B_2 t \right\} \]

\( B_1 \) and \( B_2 \) are constants that can be determined from initial conditions. The motion is no longer harmonic as shown in the figure.
The system returns to the equilibrium position in short time.
The shape of the curve depends on initial conditions as shown.
Solution # 3: over-damped vibration ($\zeta > 1$)
For this case the roots of the characteristic equation become:

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

Therefore, the solution can be written as,

$$x(t) = B_1 e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega nt} + B_2 e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega nt}$$

$B_1$ and $B_2$ are constants to be determined from knowing the initial conditions of the motion.
Graphical representation of the motions of the damped systems.
Single DoF free vibration system

Phase plane of a damped system.
Logarithmic decrement

The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes.
Logarithmic decrement:

\[
\frac{x_1(t)}{x_2(t)} = \frac{A e^{-\zeta \omega_n t_1} \cos(\omega_d t_1 - \phi)}{A e^{-\zeta \omega_n t_2} \cos(\omega_d t_2 - \phi)}
\]

So

\[
\frac{x_1(t)}{x_2(t)} = \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n (t_1 + \tau_d)}} = e^{\zeta \omega_n \tau_d}
\]

But

\[
t_2 = t_1 + \tau_d \Rightarrow \tau_d = \frac{2\pi}{\omega_d}
\]

\[
\cos(\omega_d t_2 - \phi) = \cos(2\pi + \omega_d t_1 - \phi) = \cos(\omega_d t_1 - \phi)
\]

Assume: \(\delta\) is the logarithmic decrement

\[
\delta = \ln \frac{x_1(t)}{x_2(t)} = \zeta \omega_n \tau_d = \zeta \omega_n \frac{2\pi}{\sqrt{1 - \zeta^2} \omega_n} = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}
\]

Logarithmic decrement: is dimensionless
Logarithmic decrement

\[ \delta = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \]

For small damping; \( \zeta << 1 \)

\[ \delta \approx 2\pi\zeta \]
Generally, when the amplitude after a number of cycles “n” is known, logarithmic decrement can be obtained as follows:

\[
\frac{X_1}{X_n} = \frac{X_1}{X_2} \times \frac{X_2}{X_3} \times \ldots \times \frac{X_{n-1}}{X_n}
\]

\[
\ln\left(\frac{X_1}{X_n}\right) = \ln\left(\frac{X_1}{X_2} \times \frac{X_2}{X_3} \times \ldots \times \frac{X_{n-1}}{X_n}\right)
\]

\[
= \ln\left(\frac{X_1}{X_2}\right) + \ln\left(\frac{X_2}{X_3}\right) + \ldots + \ln\left(\frac{X_{n-1}}{X_n}\right)
\]

\[
= \delta + \delta + \ldots + \delta = (n-1)\delta
\]

\[
\ln\left(\frac{x_1}{x_n}\right) = (n-1)\delta = (n-1)\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}
\]
Example 2.6

An under-damped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig.(a)). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig.(b).

Requirements:
1. Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2s and the amplitude $x_1$ is to be reduced to one-fourth in one half cycle (i.e. $x_{1.5} = x_1/4$ ).
Example 2.6
Note that this system is under-damped system
Solution:
Finding k and c

Here, \( n = 0.5 \) and \( \frac{x_1}{x_n} = 4 \), then the logarithmic decrement is:

\[
\delta = \frac{1}{n} \ln \left( \frac{x_1}{x_n} \right) = \frac{1}{0.5} \ln(4) = 2.7726 = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}
\]

\( \Rightarrow \zeta = 0.4037 \)

\[
\tau_d = 2 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \Rightarrow \omega_n = \frac{2\pi}{\omega_n \sqrt{1 - 0.4037^2}} = 3.4338 \text{ rad} / \text{s}
\]
The critical damping can be found as:

\[ c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N.sec/m} \]

Damping coefficient \( C \) and stiffness \( K \) can be found as:

\[ c = \zeta c_c = (0.4037)(1373.54) = 554.4981 \text{ N.sec/m} \ldots \text{(Ans.)} \]

\[ k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m} \ldots \text{(Ans.)} \]