



Mechanical Vibrations

Chapter 13

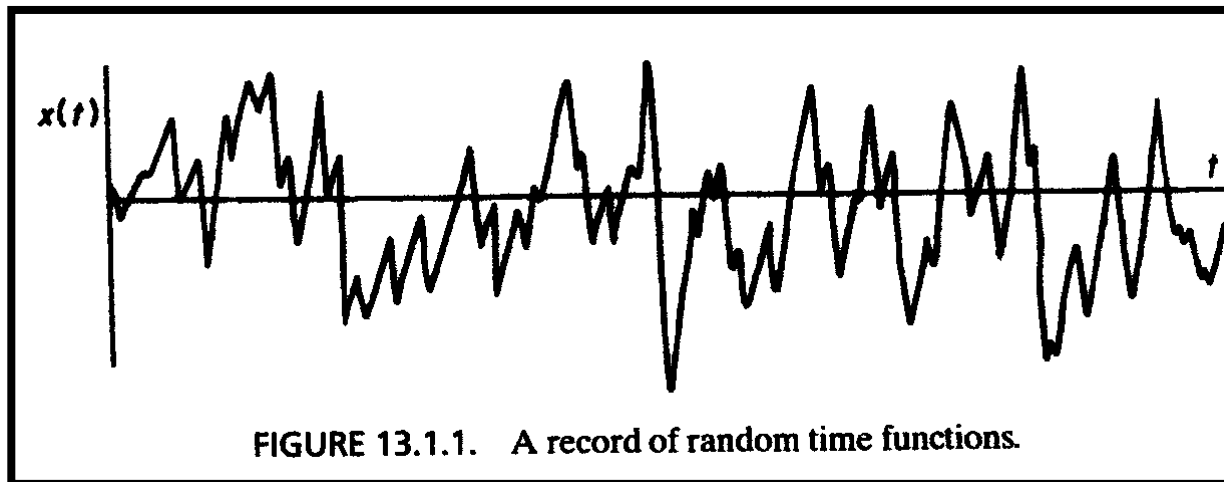
Peter Avitabile
Mechanical Engineering Department
University of Massachusetts Lowell



Random Vibrations

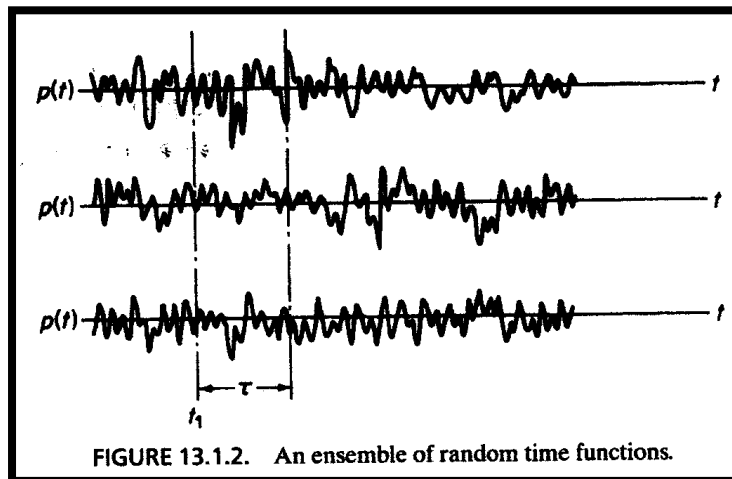
Up until now, everything analyzed was deterministic. Other loading conditions exist that are not deterministic - random vibrations.

The function below is irregular but may have some statistical character.



Random Vibrations

Each record is called a sample - the total set is called an ensemble.



If the function is evaluated at (t) and $(t + \tau)$ and the averaged the function shows no difference then the signal is stationary.



Time Average - Expected Values

The expected value can be obtained if time averaging is performed over a long time record

$$E[x(t)] = \overline{x(t)} = \langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt \quad (13.2.1)$$

Mean value $E[x^2(t)] = \overline{x^2} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt \quad (13.2.4)$

Variance $\sigma^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x - \bar{x})^2 dt = \overline{x^2} - (\bar{x})^2 \quad (13.2.5)$



Frequency Response Function

The linear input-output relationship also holds true for random signals.

In the time domain, the response can be determined in terms of the impulse response function using the convolution (Duhamel) integral as

$$x(t) = \int_0^t f(\xi)h(t - \xi)d\xi \quad (13.3.1)$$



Frequency Response Function

For the input-output problem, a simpler approach utilizes a frequency domain & frequency response function under stationary or steady state condition

For a sinusoidal excitation, the SDOF response is

$$x(t) = \frac{1}{k - m\omega^2 + jc\omega} F e^{j\omega t} \quad (13.3.4)$$

or

$$x(t) = H(\omega) F e^{j\omega t}$$

Mean Squared Response is

$$\overline{x^2} = |H(\omega)|^2 \overline{F^2}$$



Probability Distribution

Many times it is desirable to know the probability of a certain value of a time signal.

The probability density function is
$$p(x) = \lim_{\Delta x \rightarrow 0} \frac{P(x + \Delta x) - P(x)}{\Delta x} \quad (13.4.2)$$

and the variance
$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \bar{x})^2 p(x) dx = \overline{x^2} - (\bar{x})^2 \quad (13.4.7)$$

Gaussian and Rayleigh distributions widely used

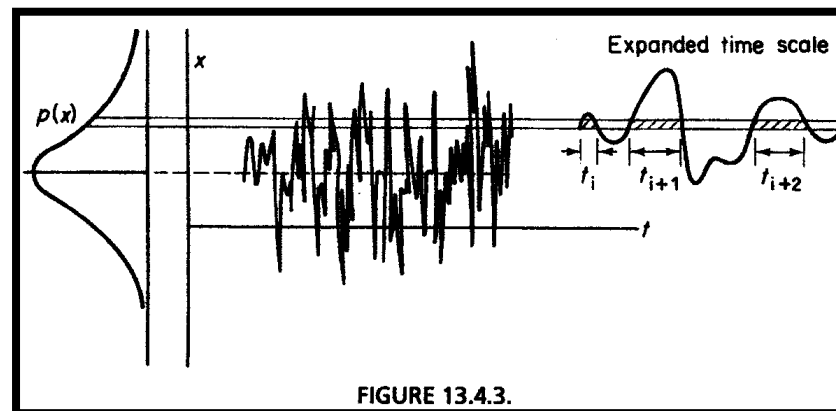
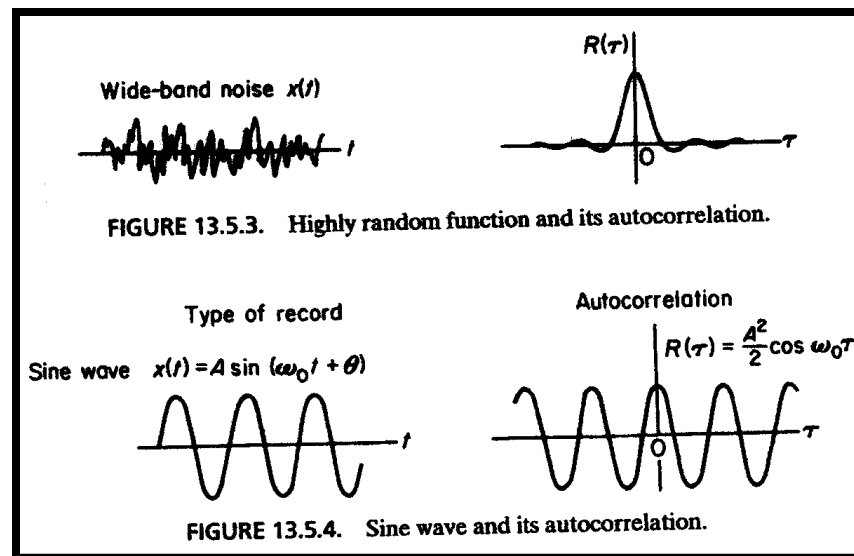


FIGURE 13.4.3.



Time Correlation Functions

Correlation is the measure of similarity between two signals. By time shifting one time signal relative to another time signal, a correlation function can be obtained.



Auto- and Cross-Correlation Function

Auto-Correlation

$$R_{xx}(\tau) = E[x(t)x(t+\tau)] = \langle x(t)x(t+\tau) \rangle$$
$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt \quad (13.5.1)$$

Cross-Correlation

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = \langle x(t)y(t+\tau) \rangle$$
$$R_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t+\tau)dt \quad (13.5.3)$$



Fourier Transforms

Fourier Integral is used for the transformation

Fourier Transform Pair

$$x(t) = \int_{-\infty}^{+\infty} S(f) e^{j2\pi ft} df \quad (13.7.1)$$

$$S(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \quad (13.7.2)$$

or using ω

Fourier Transform Pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) e^{j\omega t} d\omega \quad (13.7.3)$$

$$S(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad (13.7.4)$$



Note: Thompson uses $X(f)$ as a linear spectrum and $S(f)$ as a power spectrum
These notes use $S(f)$ as a linear spectrum and $G(f)$ as a power spectrum



Fourier Transforms

Differentiation is simply

$$\dot{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} j\omega S(\omega) e^{j\omega t} d\omega$$

which is just multiplication by $j\omega$

$$\text{FT}[\dot{x}(t)] = j\omega \text{FT}[x(t)]$$

$$\text{FT}[\ddot{x}(t)] = -\omega^2 \text{FT}[x(t)]$$



Fourier Transforms

Thus transforming the differential equation

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$(-m\omega^2 + jc\omega + k)X(\omega) = F(\omega)$$

$$H^{-1}(\omega)X(\omega) = F(\omega)$$

or as more commonly written

$$X(\omega) = H(\omega)F(\omega)$$

Note the simple multiplication rather than the convolution integral in the time domain



Parseval's Theorem

Useful for converting time domain integration into frequency domain integration

$$\begin{aligned}\int_{-\infty}^{+\infty} x_1(t)x_2(t)dt &= \int_{-\infty}^{+\infty} S_1(f)S_2^*(f)df \\ &= \int_{-\infty}^{+\infty} S_1^*(f)S_2(f)df\end{aligned}$$



Auto-Correlation Function

The auto-correlation function is

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x(t)x(t+\tau)dt$$

Applying Parseval, rearranging terms, simplifying

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} S_x(f)S_x^*(f)e^{j2\pi ft}df \quad (13.7.9)$$

$$G_{xx}(f) = S_x(f)S_x^*(f)$$

and the inverse

$$G_{xx}(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j2\pi ft}d\tau \quad (13.7.10)$$



These are the Wiener-Khintchine Equations

Cross-Correlation Function

The cross-correlation function is

$$R_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t + \tau)dt$$

Applying Parseval, rearranging terms, simplifying

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} S_x(f)S_y^*(f)e^{j2\pi ft}df \quad (13.7.12)$$

$$G_{xy}(f) = S_x(f)S_y^*(f)$$

and the inverse

$$G_{xy}(f) = \int_{-\infty}^{+\infty} R_{xy}(\tau)e^{-j2\pi ft}d\tau \quad (13.7.13)$$



These are the Wiener-Khintchine Equations

Fourier Response Technique

The Frequency Response Function (FRF) is the input-output relation

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{\text{FT}[x(t)]}{\text{FT}[f(t)]} \quad (13.8.1)$$

Multiplying and dividing by the conjugate of the input force spectrum yields

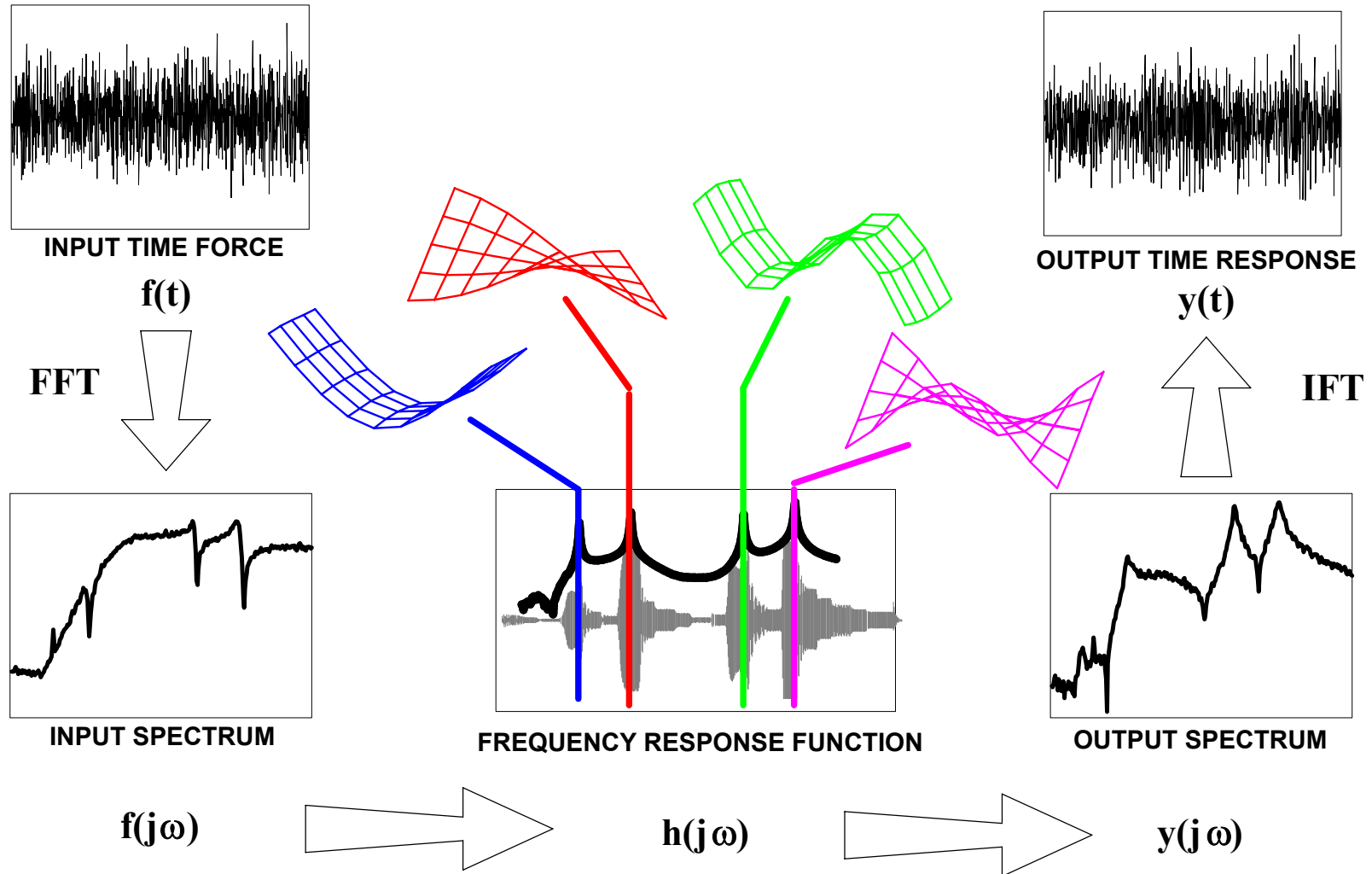
$$H(\omega) = \frac{S_x(\omega)S_f^*(\omega)}{S_f(\omega)S_f^*(\omega)} = \frac{G_{xf}(\omega)}{G_{ff}(\omega)} \quad (13.8.2)$$

$$S_x(\omega)S_f^*(\omega) = H(\omega)S_f(\omega)S_f^*(\omega)$$

$$G_{xf}(\omega) = H(\omega)G_{ff}(\omega)$$



Fourier Response Technique Schematic



Fourier Response Technique

Using the frequency domain input-output relationships,

$$\boxed{\text{Output Response} = \text{System Characteristic} \times \text{Input Forces}}$$

the response due to many forces can be computed

$$y_i(j\omega) = \sum_{j=1}^{N_0} h_{ij}(j\omega) f_j(j\omega)$$

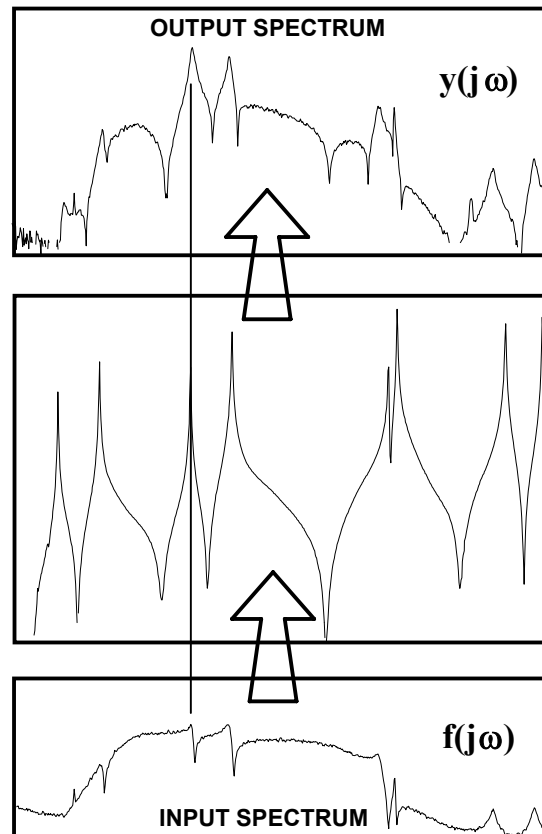
The frequency response function is needed for this response

$$h_{ij}(j\omega) = \sum_{k=1}^m \left[\frac{r_{ij,k}}{j\omega - \lambda_k} + \frac{r_{ij,k}^*}{j\omega - \lambda_k^*} \right]$$

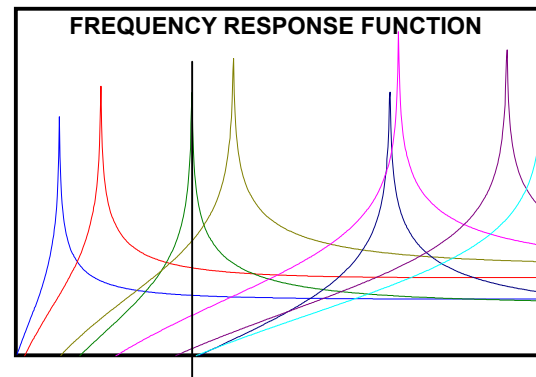


Fourier Response Technique

Frequency domain input-output schematic



$$h_{ij}(j\omega) = \sum_{k=1}^m \left[\frac{r_{ij,k}}{j\omega - p_k} + \frac{r_{ij,k}^*}{j\omega - p_k^*} \right]$$



$$y_i(j\omega) = \sum_{j=1}^{N_o} h_{ij}(j\omega) f_j(j\omega)$$

