

CHAPTER 13

Random Vibrations

The types of functions we have considered up to now can be classified as deterministic, i.e., mathematical expressions can be written that will determine their instantaneous values at any time t . There are, however, a number of physical phenomena that result in nondeterministic data for which future instantaneous values cannot be predicted in a deterministic sense. As examples, we can mention the noise of a jet engine, the heights of waves in a choppy sea, ground motion during an earthquake, and pressure gusts encountered by an airplane in flight. These phenomena all have one thing in common: the unpredictability of their instantaneous value at any future time. Nondeterministic data of this type are referred to as *random time functions*.

3.1 RANDOM PHENOMENA

A sample of a typical random time function is shown in Fig. 13.1.1. In spite of the irregular character of the function, many random phenomena exhibit some degree of statistical regularity, and certain averaging procedures can be applied to establish gross characteristics useful in engineering design.

In any statistical method, a large amount of data is necessary to establish reliability. For example, to establish the statistics of the pressure fluctuation due to air turbulence over a certain air route, an airplane may collect hundreds of records of the type shown in Fig. 13.1.2.

Each record is called a *sample*, and the total collection of samples is called the *ensemble*. We can compute the ensemble average of the instantaneous pressures in each sample at time t_1 . We can also multiply the instantaneous pressures in each sample at times t_1 and $t_1 + \tau$, and average these results for the ensemble. If such averages do not differ as we choose different values of t_1 , then the random process described by this ensemble is said to be *stationary*.

If the ensemble averages are replaced next by time averages, and if the results computed from each sample are the same as those of any other sample and equal to the ensemble average, then the random process is said to be *ergodic*.

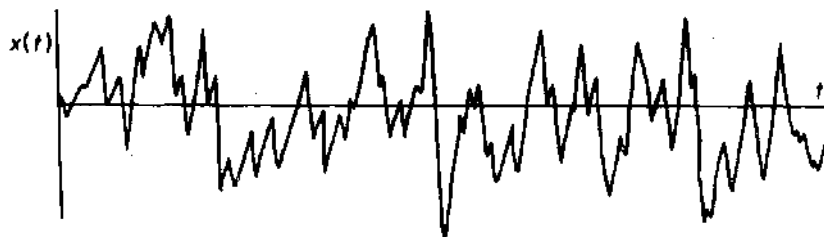


FIGURE 13.1.1. A record of random time functions.

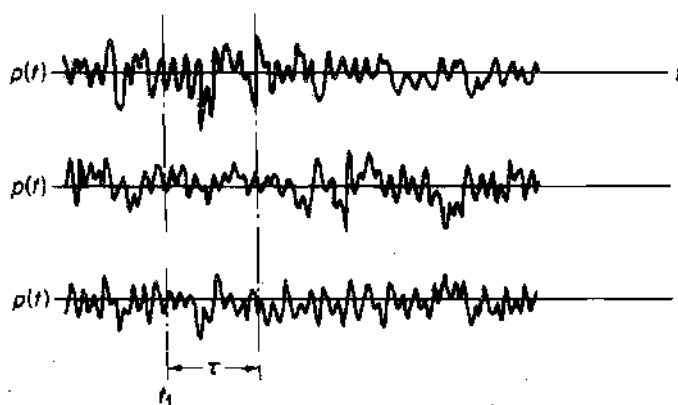


FIGURE 13.1.2. An ensemble of random time functions.

Thus, for a stationary ergodic random phenomenon, its statistical properties are available from a single time function of a sufficiently long time period. Although such random phenomena may exist only theoretically, its assumption greatly simplifies the task of dealing with random variables. This chapter treats only this class of stationary ergodic random functions.

13.2 TIME AVERAGING AND EXPECTED VALUE

Expected value. In random vibrations, we repeatedly encounter the concept of time averaging over a long period of time. The most common notation for this operation is defined by the following equation in which $x(t)$ is the variable.

$$\overline{x(t)} = \langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt \quad (13.2.1)$$

This number is also equal to the *expected value* of $x(t)$, which is written as

$$E[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt \quad (13.2.2)$$

It is the average or mean value of a quantity sampled over a long time. In the case of discrete variables x_i , the expected value is given by the equation

$$E[x] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i \quad (13.2.3)$$

Mean square value. These average operations can be applied to any variable such as $x^2(t)$ or $x(t) \cdot y(t)$. The *mean square value*, designated by the notation $\bar{x}^2$ or $E[x^2(t)]$, is found by integrating $x^2(t)$ over a time interval T and taking its average value according to the equation

$$E[x^2(t)] = \bar{x}^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2 dt \quad (13.2.4)$$

Variance and standard deviation. It is often desirable to consider the time series in terms of the mean and its fluctuation from the mean. A property of importance describing the fluctuation is the *variance* σ^2 , which is the mean square value about the mean, given by the equation

$$\sigma^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x - \bar{x})^2 dt \quad (13.2.5)$$

By expanding the above equation, it is easily seen that

$$\sigma^2 = \bar{x}^2 - (\bar{x})^2 \quad (13.2.6)$$

so that the variance is equal to the mean square value minus the square of the mean. The positive square root of the variance is the *standard deviation*, σ .

Fourier series. Generally, random time functions contain oscillations of many frequencies, which approach a continuous spectrum. Although random time functions are generally not periodic, their representations by Fourier series, in which the periods are extended to a large value approaching infinity, offers a logical approach.

In Chapter 1, the exponential form of the Fourier series was shown to be

$$x(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega_1 t} = c_0 + \sum_{n=1}^{\infty} (c_n e^{in\omega_1 t} + c_n^* e^{-in\omega_1 t}) \quad (13.2.7)$$

This series, which is a real function, involves a summation over negative and positive frequencies, and it also contains a constant term c_0 . The constant term c_0 is the average value of $x(t)$ and because it can be dealt with separately, we exclude it in future considerations. Moreover, actual measurements are made in terms of positive frequencies, and it would be more desirable to work with the equation

$$x(t) = \text{Re} \sum_{n=1}^{\infty} C_n e^{in\omega_1 t} \quad (13.2.8)$$

The one-sided summation in the previous equation is complex and, hence, the real part of the series must be stipulated for $x(t)$ real. Because the real part of a vector is one-half the sum of the vector and its conjugate [see Eq. (1.1.9)],

$$x(t) = \operatorname{Re} \sum_{n=1}^{\infty} C_n e^{in\omega_0 t} = \frac{1}{2} \sum_{n=1}^{\infty} (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t})$$

By comparison with Eq. (1.2.6), we find

$$\begin{aligned} C_n &= 2c_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) e^{-in\omega_0 t} dt \\ &= a_n - ib_n \end{aligned} \quad (13.2.9)$$

EXAMPLE 13.2.1

Determine the mean square value of a record of random vibration $x(t)$ containing many discrete frequencies.

Solution Because the record is periodic, we can represent it by the real part of the Fourier series:

$$\begin{aligned} x(t) &= \operatorname{Re} \sum_{n=1}^{\infty} C_n e^{in\omega_0 t} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t}) \end{aligned}$$

where C_n is a complex number, and C_n^* is its complex conjugate. [See Eq. (13.2.9).] Its mean square value is

$$\begin{aligned} x^2 &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{1}{4} \sum_{n=1}^{\infty} (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t})^2 dt \\ &= \lim_{T \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{C_n^2 e^{i2n\omega_0 t}}{i2n\omega_0 T} + 2C_n C_n^* + \frac{C_n^{*2} e^{-i2n\omega_0 t}}{-i2n\omega_0 T} \right) \bigg|_0^T \\ &= \sum_{n=1}^{\infty} \frac{1}{2} C_n C_n^* = \sum_{n=1}^{\infty} \frac{1}{2} |C_n|^2 = \sum_{n=1}^{\infty} \overline{C_n^2} \end{aligned}$$

In this equation, $e^{\pm i2n\omega_0 t}$, for any t , is bounded between ± 1 , and due to $T \rightarrow \infty$ in the denominator, the first and last terms become zero. The middle term, however, is independent of T . Thus, the mean square value of the periodic function is simply the sum of the mean square value of each harmonic component present.

13.3 FREQUENCY RESPONSE FUNCTION

In any linear system, there is a direct linear relationship between the input and the output. This relationship, which also holds for random functions, is represented by the block diagram of Fig. 13.3.1.

In the time domain, the system behavior can be determined in terms of the system impulse response $h(t)$ used in the convolution integral of Eq. (4.2.1).

$$y(t) = \int_0^t x(\xi) h(t - \xi) d\xi \quad (13.3.1)$$

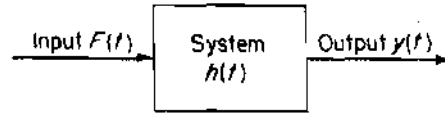


FIGURE 13.3.1. Input-output relationship of a linear system.

A much simpler relationship is available for the frequency domain in terms of the frequency response function $H(\omega)$, which we can define as the ratio of the output to the input under steady-state conditions, with the input equal to a harmonic time function of unit amplitude. The transient solution is thus excluded in this consideration. In random vibrations, the initial conditions and the phase have little meaning and are therefore ignored. We are mainly concerned with the average energy, which we can associate with the mean square value.

Applying this definition to a single-DOF system,

$$m\ddot{y} + c\dot{y} + ky = x(t) \quad (13.3.2)$$

let the input be $x(t) = e^{i\omega t}$. The steady-state output will then be $y = H(\omega)e^{i\omega t}$, where $H(\omega)$ is a complex function. Substituting these into the differential equation and canceling $e^{i\omega t}$ from each side, we obtain

$$(-m\omega^2 + ic\omega + k)H(\omega) = 1$$

The frequency response function is then

$$\begin{aligned} H(\omega) &= \frac{1}{k - m\omega^2 + ic\omega} \\ &= \frac{1}{k} \frac{1}{1 - (\omega/\omega_n)^2 + i2\zeta(\omega/\omega_n)} \end{aligned} \quad (13.3.3)$$

As mentioned in Chapter 3, we will absorb the factor $1/k$ in with the force. $H(\omega)$ is then a nondimensional function of ω/ω_n and the damping factor ζ .

The input-output relationship in terms of the frequency-response function can be written as

$$y(t) = H(\omega)F_0 e^{i\omega t} \quad (13.3.4)$$

where $F_0 e^{i\omega t}$ is a harmonic function.

For the mean square response, we follow the procedure of Example 13.2.1 and write

$$y = \frac{1}{2}F_0(He^{i\omega t} + H^*e^{-i\omega t}) \quad (13.3.5)$$

Thus, by squaring and substituting into Eq. (13.2.4), we find the mean square value of y is

$$\begin{aligned} \overline{y^2} &= \frac{F_0^2}{4} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (H^2 e^{i2\omega t} + 2HH^* + H^*e^{-i2\omega t}) dt \\ &= \frac{F_0^2}{2} H(\omega)H^*(\omega) = \overline{F_0^2} |H(\omega)|^2 \end{aligned} \quad (13.3.6)$$

In the preceding equation, the first and last terms become zero because of $T \rightarrow \infty$ in the denominator, whereas the middle term is independent of T . Equation (13.3.6) indicates that the mean square value of the response is equal to the mean square excitation multiplied by the square of the absolute values of the frequency response function. For excitations expressed in terms of Fourier series with many frequencies, the response is the sum of terms similar to Eq. (13.3.6).

EXAMPLE 13.3.1

A single-DOF system with natural frequency $\omega_n = \sqrt{k/m}$ and damping $\zeta = 0.20$ is excited by the force

$$F(t) = F \cos \frac{1}{2} \omega_n t + F \cos \omega_n t + F \cos \frac{3}{2} \omega_n t$$

$$= \sum_{m=1/2, 1, 3/2} F \cos m \omega_n t$$

Determine the mean square response and compare the output spectrum with that of the input.

Solution The response of the system is simply the sum of the response of the single-DOF system to each of the harmonic components of the exciting force.

$$x(t) = \sum_{m=1/2, 1, 3/2} |H(m\omega)| F \cos (m\omega_n t + \phi_m)$$

where

$$|H(\frac{1}{2}\omega_n)| = \frac{1/k}{\sqrt{9/16 + (0.20)^2}} = \frac{1.29}{k}$$

$$|H(\omega_n)| = \frac{1/k}{\sqrt{4(0.20)^2}} = \frac{2.50}{k}$$

$$|H(\frac{3}{2}\omega_n)| = \frac{1/k}{\sqrt{25/16 + 9(0.20)^2}} = \frac{0.72}{k}$$

$$\phi_{1/2} = \tan^{-1} \frac{4\zeta}{3} = 0.083\pi$$

$$\phi_1 = \tan^{-1} \infty = 0.50\pi$$

$$\phi_{3/2} = \tan^{-1} \frac{-12\zeta}{5} = -0.142\pi$$

Substituting these values into $x(t)$, we obtain the equation

$$x(t) = \frac{F}{k} [1.29 \cos (0.5\omega_n t - 0.083\pi) \\ + 2.50 \cos (\omega_n t - 0.50\pi) \\ + 0.72 \cos (1.5\omega_n t + 0.142\pi)]$$

The mean square response is then

$$\overline{x^2} = \frac{F^2}{2k^2} [(1.29)^2 + (2.50)^2 + (0.72)^2]$$

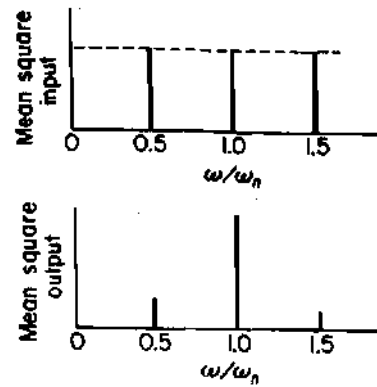


FIGURE 13.3.2. Input and output spectra with discrete frequencies.

Figure 13.3.2 shows the input and output spectra for the problem. The components of the mean square input are the same for each frequency and equal to $F^2/2$. The output spectrum is modified by the system frequency-response function.

13.4 PROBABILITY DISTRIBUTION

By referring to the random time function of Fig. 13.4.1, what is the probability of its instantaneous value being less than (more negative than) some specified value x_1 ? To answer this question, we draw a horizontal line at the specified value x_1 and sum the time intervals Δt_i during which $x(t)$ is less than x_1 . This sum divided by the total time then represents the fraction of the total time that $x(t)$ is less than x_1 , which is the probability that $x(t)$ will be found less than x_1 .

$$P(x_1) = \text{Prob}[x(t) < x_1] \\ = \lim_{t \rightarrow \infty} \frac{1}{t} \sum \Delta t_i \quad (13.4.1)$$

If a large negative number is chosen for x_1 , none of the curve will extend negatively beyond x_1 , and, hence, $P(x_1 \rightarrow -\infty) = 0$. As the horizontal line corresponding to x_1 is moved up, more of $x(t)$ will extend negatively beyond x_1 , and the fraction of the total

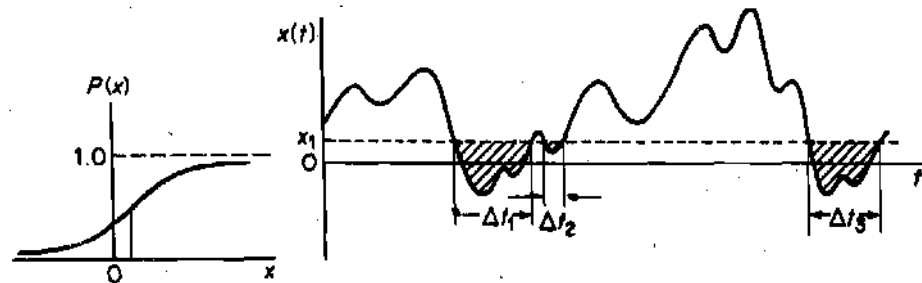


FIGURE 13.4.1. Calculation of cumulative probability.

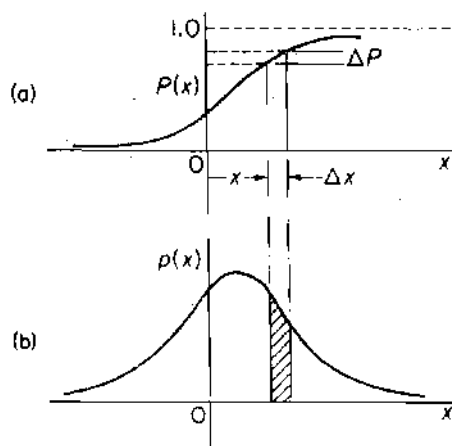


FIGURE 13.4.2. (a) Cumulative probability, (b) Probability density.

time in which $x(t)$ extends below x_1 must increase, as shown in Fig. 13.4.2(a). As $x \rightarrow \infty$, all $x(t)$ will lie in the region less than $x = \infty$, and, hence, the probability of $x(t)$ being less than $x = \infty$ is certain, or $P(x = \infty) = 1.0$. Thus, the curve of Fig. 13.4.2(a), which is cumulative toward positive x , must increase monotonically from 0 at $x = -\infty$ to 1.0 at $x = +\infty$. The curve is called the cumulative probability distribution function $P(x)$.

If next we wish to determine the probability of $x(t)$ lying between the values x_1 and $x_1 + \Delta x$, all we need to do is subtract $P(x_1)$ from $P(x_1 + \Delta x)$, which is also proportional to the time occupied by $x(t)$ in the zone x_1 to $x_1 + \Delta x$.

We now define the *probability density function* $p(x)$ as

$$p(x) = \lim_{\Delta x \rightarrow 0} \frac{P(x + \Delta x) - P(x)}{\Delta x} = \frac{dP(x)}{dx} \quad (13.4.2)$$

and it is evident from Fig. 13.4.2(b) that $p(x)$ is the slope of the cumulative probability distribution $P(x)$. From the preceding equation, we can also write

$$P(x_1) = \int_{-\infty}^{x_1} p(x) dx \quad (13.4.3)$$

The area under the probability density curve of Fig. 13.4.2(b) between two values of x represents the probability of the variable being in this interval. Because the probability of $x(t)$ being between $x = \pm\infty$ is certain,

$$P(\infty) = \int_{-\infty}^{+\infty} p(x) dx = 1.0 \quad (13.4.4)$$

and the total area under the $p(x)$ curve must be unity. Figure 13.4.3 again illustrates the probability density $p(x)$, which is the fraction of the time occupied by $x(t)$ in the interval x to $x + dx$.

The mean and the mean square value, previously defined in terms of the time average, are related to the probability density function in the following manner. The

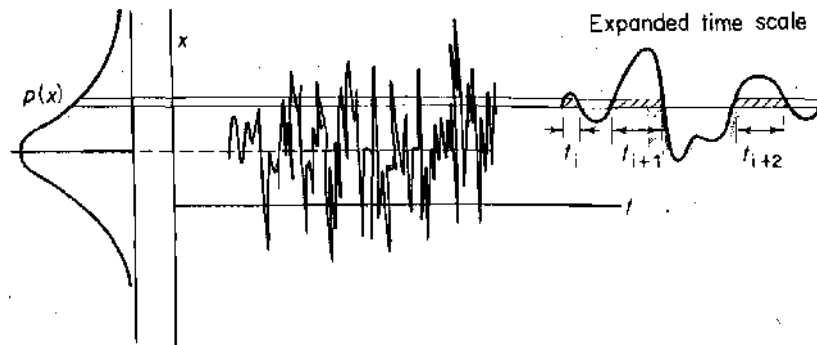
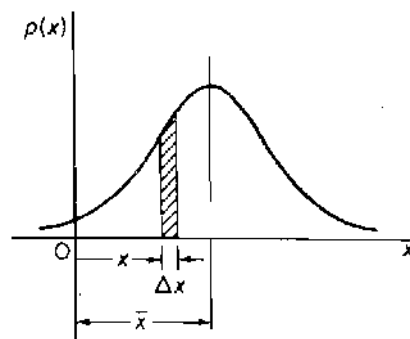


FIGURE 13.4.3.


 FIGURE 13.4.4. First and second moments of $p(x)$.

mean value $\bar{x}$ coincides with the centroid of the area under the probability density curve $p(x)$, as shown in Fig. 13.4.4. Therefore, it can be determined by the first moment:

$$\bar{x} = \int_{-\infty}^{\infty} xp(x) dx \quad (13.4.5)$$

Likewise, the mean square value is determined from the second moment

$$\bar{x}^2 = \int_{-\infty}^{\infty} x^2 p(x) dx \quad (13.4.6)$$

which is analogous to the moment of inertia of the area under the probability density curve about $x = 0$.

The *variance* σ^2 , previously defined as the mean square value about the mean, is

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x - \bar{x})^2 p(x) dx \\ &= \int_{-\infty}^{\infty} x^2 p(x) dx - 2\bar{x} \int_{-\infty}^{\infty} xp(x) dx + (\bar{x})^2 \int_{-\infty}^{\infty} p(x) dx \\ &= \bar{x}^2 - 2(\bar{x})^2 + (\bar{x})^2 \\ &= \bar{x}^2 - (\bar{x})^2 \end{aligned} \quad (13.4.7)$$

The *standard deviation* σ is the positive square root of the variance. When the mean value is zero, $\sigma = \sqrt{\overline{x^2}}$, and the standard deviation is equal to the root-mean-square (rms) value.

Gaussian and Rayleigh distributions. Certain distributions that occur frequently in nature are the *Gaussian* (or normal) distribution and the *Rayleigh* distribution, both of which can be expressed mathematically. The Gaussian distribution is a bell-shaped curve, symmetric about the mean value (which will be assumed to be zero) with the following equation:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \quad (13.4.8)$$

The standard deviation σ is a measure of the spread about the mean value; the smaller the value of σ , the narrower the $p(x)$ curve (remember that the total area = 1.0), as shown in Fig. 13.4.5(a).

In Fig. 13.4.5(b), the Gaussian distribution is plotted nondimensionally in terms of x/σ . The probability of $x(t)$ being between $\pm\lambda\sigma$, where λ is any positive number, is found from the equation

$$\text{Prob} [-\lambda\sigma \leq x(t) \leq \lambda\sigma] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\lambda\sigma}^{\lambda\sigma} e^{-x^2/2\sigma^2} dx \quad (13.4.9)$$

The following table presents numerical values associated with $\lambda = 1, 2$, and 3.

λ	Prob $[-\lambda\sigma \leq x(t) \leq \lambda\sigma]$	Prob $[x > \lambda\sigma]$
1	68.3%	31.7%
2	95.4%	4.6%
3	99.7%	0.3%

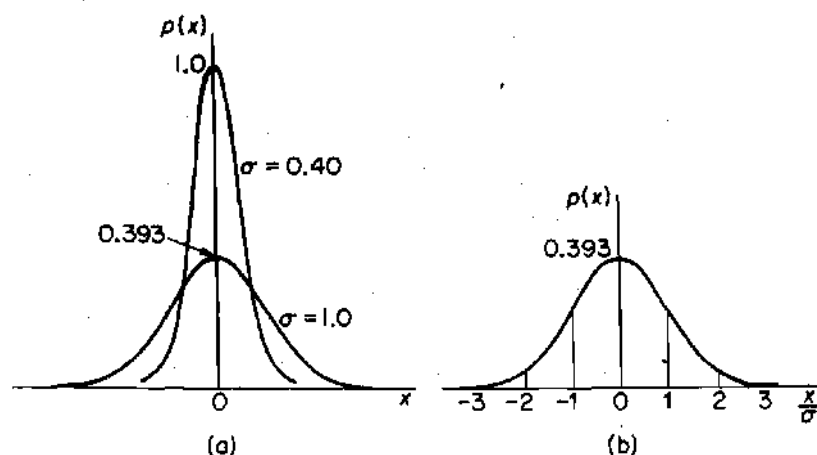


FIGURE 13.4.5. Normal distribution.

The probability of $x(t)$ lying outside $\pm \lambda \sigma$ is the probability of $|x|$ exceeding $\lambda \sigma$, which is 1.0 minus the preceding values, or the equation

$$\text{Prob } [|x| > \lambda \sigma] = \frac{2}{\sigma \sqrt{2\pi}} \int_{\lambda \sigma}^{\infty} e^{-x^2/2\sigma^2} dx = \text{erfc}\left(\frac{\lambda}{\sqrt{2}}\right) \quad (13.4.10)$$

Random variables restricted to positive values, such as the absolute value A of the amplitude, often tend to follow the Rayleigh distribution, which is defined by the equation

$$p(A) = \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} \quad A > 0 \quad (13.4.11)$$

The probability density $p(A)$ is zero here for $A < 0$ and has the shape shown in Fig. 13.4.6.

The mean and mean square values for the Rayleigh distribution can be found from the first and second moments to be

$$\begin{aligned} \bar{A} &= \int_0^{\infty} A p(A) dA = \int_0^{\infty} \frac{A^2}{\sigma^2} e^{-A^2/2\sigma^2} dA = \sqrt{\frac{\pi}{2}} \sigma \\ \bar{A}^2 &= \int_0^{\infty} A^2 p(A) dA = \int_0^{\infty} \frac{A^3}{\sigma^2} e^{-A^2/2\sigma^2} dA = 2\sigma^2 \end{aligned} \quad (13.4.12)$$

The variance associated with the Rayleigh distribution is

$$\begin{aligned} \sigma_A^2 &= \bar{A}^2 - (\bar{A})^2 = \left(\frac{4 - \pi}{2}\right) \sigma^2 \\ \therefore \sigma_A &\equiv \frac{2}{3} \sigma \end{aligned} \quad (13.4.13)$$

Also, the probability of A exceeding a specified value $\lambda \sigma$ is

$$\text{Prob } [A > \lambda \sigma] = \int_{\lambda \sigma}^{\infty} \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} dA \quad (13.4.14)$$

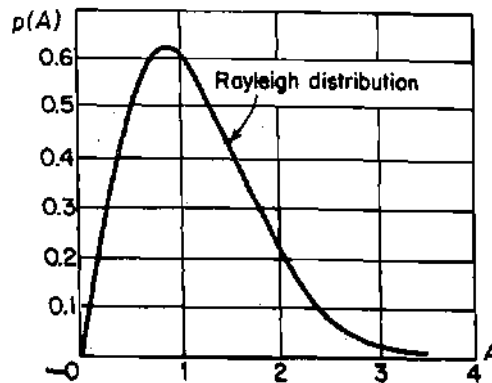


FIGURE 13.4.6. Rayleigh distribution.

which has the following numerical values:

λ	$P[A > \lambda\sigma]$
0	100%
1	60.7%
2	13.5%
3	1.2%

Three important examples of time records frequently encountered in practice are shown in Fig. 13.4.7, where the mean value is arbitrarily chosen to be zero. The cumulative probability distribution for the sine wave is easily shown to be

$$P(x) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1} \frac{x}{A}$$

and its probability density, by differentiation, is

$$p(x) = \frac{1}{\pi \sqrt{A^2 - x^2}} \quad |x| < A$$

$$= 0 \quad |x| > A$$

For the wide-band record, the amplitude, phase, and frequency all vary randomly and an analytical expression is not possible for its instantaneous value. Such functions are encountered in radio noise, jet engine pressure fluctuation, atmospheric turbulence, and so on, and a most likely probability distribution for such records is the *Gaussian distribution*.

When a wide-band record is put through a narrow-band filter, or a resonance system in which the filter bandwidth is small compared to its central frequency f_0 , we obtain the third type of wave, which is essentially a constant-frequency oscillation with

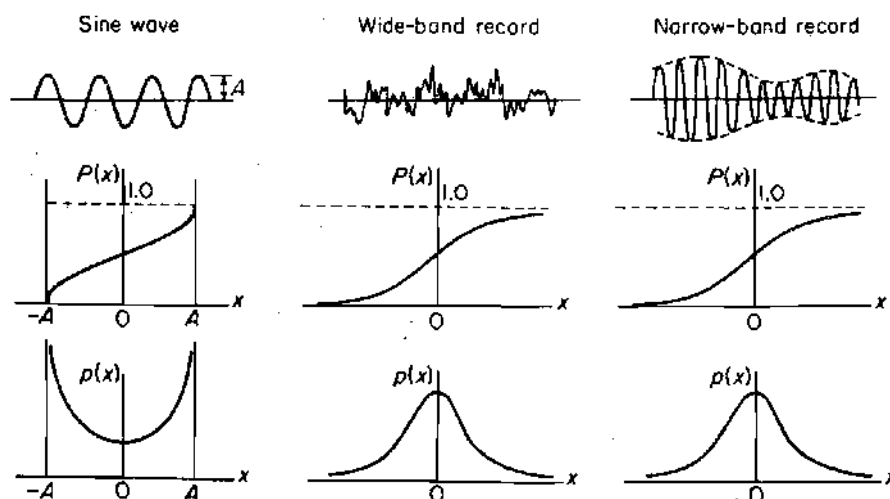


FIGURE 13.4.7. Probability functions for three types of records.

slowly varying amplitude and phase. The probability distribution for its instantaneous values is the same as that for the wide-band random function. However, the absolute values of its peaks, corresponding to the envelope, will have a Rayleigh distribution.

Another quantity of great interest is the distribution of the peak values. Rice¹ shows that the distribution of the peak values depends on a quantity $N_0/2M$, where N_0 is the number of zero crossings, and $2M$ is the number of positive and negative peaks. For a sine wave or a narrow band, N_0 is equal to $2M$, so that the ratio $N_0/2M = 1$. For a wide-band random record, the number of peaks will greatly exceed the number of zero crossings, so that $N_0/2M$ tends to approach zero. When $N_0/2M = 0$, the probability density distribution of peak values turns out to be Gaussian, whereas when $N_0/2M = 1$, as in the narrow-band case, the probability density distribution of the peak values tends to a *Rayleigh distribution*.

13.5 CORRELATION

Correlation is a measure of the similarity between two quantities. As it applies to vibration waveforms, correlation is a time-domain analysis useful for detecting hidden periodic signals buried in measurement noise, propagation time through the structure, and for determining other information related to the structure's spectral characteristics, which are better discussed under Fourier transforms.

Suppose we have two records, $x_1(t)$ and $x_2(t)$, as shown in Fig. 13.5.1. The *correlation* between them is computed by multiplying the ordinates of the two records at each time t and determining the average value $\langle x_1(t)x_2(t) \rangle$ by dividing the sum of the products by the number of products. It is evident that the correlation so found will be largest when the two records are similar or identical. For dissimilar records, some of the products will be positive and others will be negative, so their sum will be smaller.

Next, consider the case in which $x_2(t)$ is identical to $x_1(t)$ but shifted to the left by a time τ , as shown in Fig. 13.5.2. Then, at time (t) , when x_1 is $x(t)$, the value of x_2 is $x(t + \tau)$, and the correlation is given by $\langle x(t)x(t + \tau) \rangle$. Here, if $\tau = 0$, we have complete correlation. As τ increases, the correlation decreases.

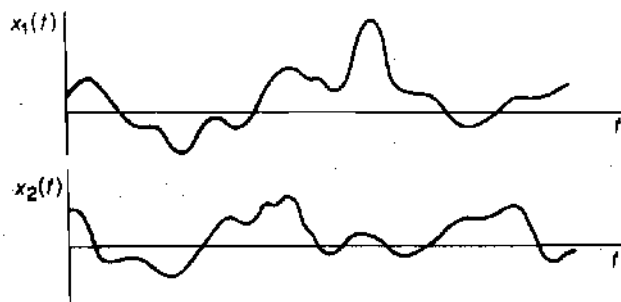
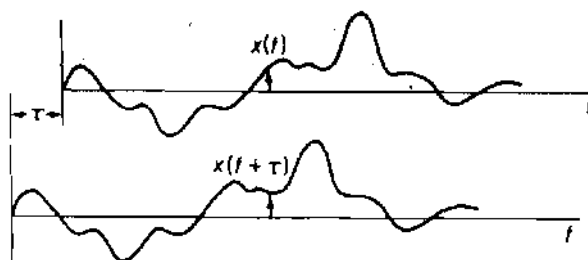


FIGURE 13.5.1. Correlation between $x_1(t)$ and $x_2(t)$.

¹See Ref. [8].

FIGURE 13.5.2. Function $x(t)$ shifted by τ .

It is evident that this result can be computed from a single record by multiplying the ordinates at time t and $t + \tau$ and determining the average. We then call this result the *autocorrelation* and designate it by $R(\tau)$. It is also the expected value of the product $x(t)x(t + \tau)$, or

$$\begin{aligned} R(\tau) &= E[x(t)x(t + \tau)] = \langle x(t)x(t + \tau) \rangle \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t + \tau) dt \end{aligned} \quad (13.5.1)$$

When $\tau = 0$, this definition reduces to the mean square value:

$$R(0) = \overline{x^2} = \sigma^2 \quad (13.5.2)$$

Because the second record of Fig. 13.5.2 can be considered to be delayed with respect to the first record, or the first record advanced with respect to the second record, it is evident that $R(\tau) = R(-\tau)$ is symmetric about the origin $\tau = 0$ and is always less than $R(0)$.

Highly random functions, such as the wide-band noise shown in Fig. 13.5.3, soon lose their similarity within a short time shift. Its autocorrelation, therefore is a sharp spike at $\tau = 0$ that drops off rapidly with $\pm \tau$ as shown. It implies that wide-band random records have little or no correlation except near $\tau = 0$.

For the special case of a periodic wave, the autocorrelation must be periodic of the same period, because shifting the wave one period brings the wave into coincidence again. Figure 13.5.4 shows a sine wave and its autocorrelation.

For the narrow-band record shown in Fig. 13.5.5, the autocorrelation has some of the characteristics found for the sine wave in that it is again an even function with a maximum at $\tau = 0$ and frequency ω_0 corresponding to the dominant or central frequency. The difference appears in the fact that $R(\tau)$ approaches zero for large τ for the narrow-band record. It is evident from this discussion that hidden periodicities in a noisy random record can be detected by correlating the record with a sinusoid. There will be almost no correlation between the sinusoid and the noise that will be suppressed. By exploring with sinusoids of differing frequencies, the hidden periodic signal can be detected. Figure 13.5.6 shows a block diagram for the determination of the autocorrelation. The signal $x(t)$ is delayed by τ and multiplied, after which it is integrated and averaged. The delay time τ is fixed during each run and is changed in steps or is continuously changed by a slow sweeping technique. If the record is on magnetic tape, the time delay τ can be accomplished by passing the tape between two identical pickup units, as shown in Fig. 13.5.7.

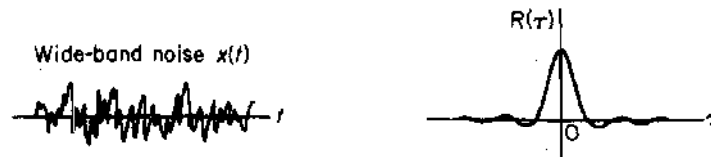


FIGURE 13.5.3. Highly random function and its autocorrelation.

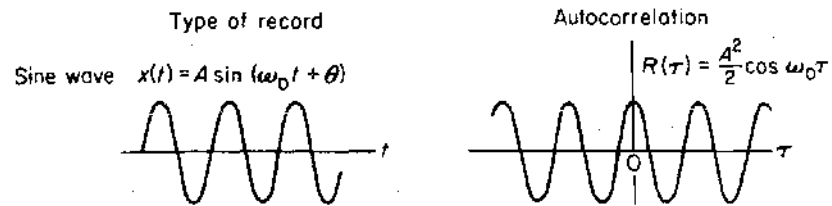


FIGURE 13.5.4. Sine wave and its autocorrelation.

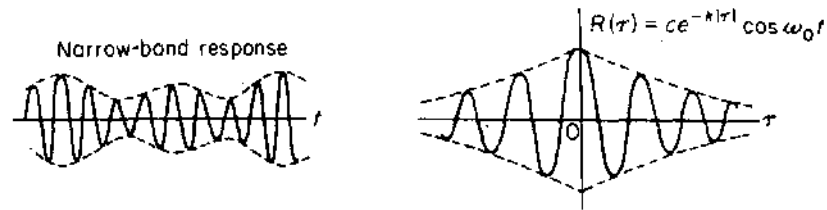


FIGURE 13.5.5. Autocorrelation for the narrow-band record.

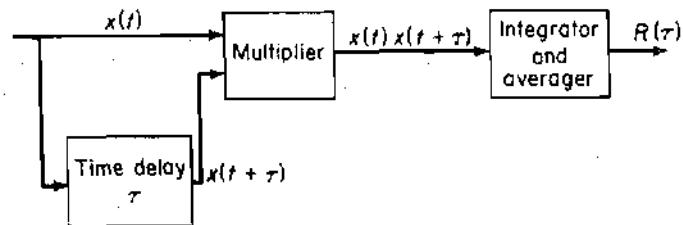


FIGURE 13.5.6. Block diagram of the autocorrelation analyzer.

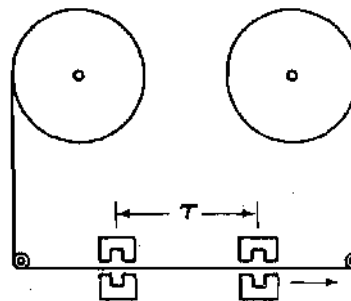


FIGURE 13.5.7. Time delay for autocorrelation.

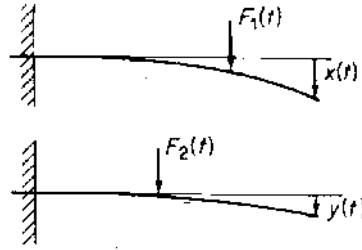


FIGURE 13.5.8.

Cross correlation. Consider two random quantities $x(t)$ and $y(t)$. The correlation between these two quantities is defined by the equation

$$\begin{aligned} R_{xy}(\tau) &= E[x(t)y(t + \tau)] = \langle x(t)y(t + \tau) \rangle \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t + \tau) dt \end{aligned} \quad (13.5.3)$$

which can also be called the *cross correlation* between the quantities x and y .

Such quantities often arise in dynamical problems. For example, let $x(t)$ be the deflection at the end of a beam due to a load $F_1(t)$ at some specified point. $y(t)$ is the deflection at the same point, due to a second load $F_2(t)$ at a different point than the first, as illustrated in Fig. 13.5.8. The deflection due to both loads is then $z(t) = x(t) + y(t)$, and the autocorrelation of $z(t)$ as a result of the two loads is

$$\begin{aligned} R_z(\tau) &= \langle z(t)z(t + \tau) \rangle \\ &= \langle [x(t) + y(t)][x(t + \tau) + y(t + \tau)] \rangle \\ &= \langle x(t)x(t + \tau) \rangle + \langle x(t)y(t + \tau) \rangle \\ &\quad + \langle y(t)x(t + \tau) \rangle + \langle y(t)y(t + \tau) \rangle \\ &= R_x(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_y(\tau) \end{aligned} \quad (13.5.4)$$

Thus, the autocorrelation of a deflection at a given point due to separate loads $F_1(t)$ and $F_2(t)$ cannot be determined simply by adding the autocorrelations $R_x(\tau)$ and $R_y(\tau)$ resulting from each load acting separately. $R_{xy}(\tau)$ and $R_{yx}(\tau)$ are here referred to as *cross correlation*, and, in general, they are not equal.

EXAMPLE 13.5.1

Show that the autocorrelation of the rectangular gating function shown in Fig. 13.5.9 is a triangle.

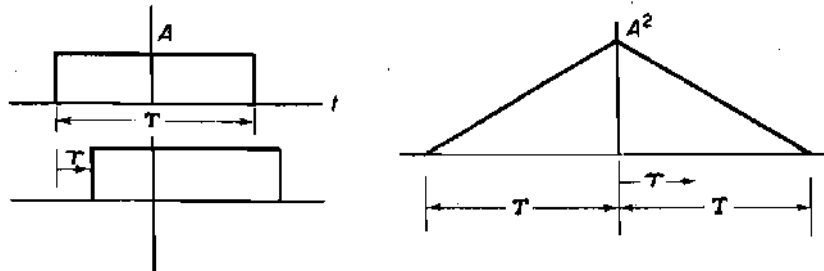


FIGURE 13.5.9. Autocorrelation of a rectangle is a triangle.

Solution If the rectangular pulse is shifted in either direction by τ , its product with the original pulse is $A^2(T - \tau)$. It is easily seen then that starting with $\tau = 0$, the autocorrelation curve is a straight line that forms a triangle with height A^2 and base equal to $2T$.

3.6 POWER SPECTRUM AND POWER SPECTRAL DENSITY

The frequency composition of a random function can be described in terms of the spectral density of the mean square value. We found in Example 13.2.1 that the mean square value of a periodic time function is the sum of the mean square value of the individual harmonic component present.

$$\overline{x^2} = \sum_{n=1}^{\infty} \frac{1}{2} C_n C_n^*$$

Thus $\overline{x^2}$ is made up of discrete contributions in each frequency interval Δf .

We first define the contribution to the mean square in the frequency interval Δf as the *power spectrum* $G(f_n)$:

$$G(f_n) = \frac{1}{2} C_n C_n^* \quad (13.6.1)$$

The mean square value is then

$$\overline{x^2} = \sum_{n=1}^{\infty} G(f_n) \quad (13.6.2)$$

We now define the *discrete power spectral density* $S(f_n)$ as the power spectrum divided by the frequency interval Δf :

$$S(f_n) = \frac{G(f_n)}{\Delta f} = \frac{C_n C_n^*}{2\Delta f} \quad (13.6.3)$$

The mean square value can then be written as

$$\overline{x^2} = \sum_{n=1}^{\infty} S(f_n) \Delta f \quad (13.6.4)$$

The power spectrum and the power spectral density will hereafter be abbreviated as PS and PSD, respectively.

An example of discrete PSD is shown in Fig. 13.6.1. When $x(t)$ contains a very large number of frequency components, the lines of the discrete spectrum become closer

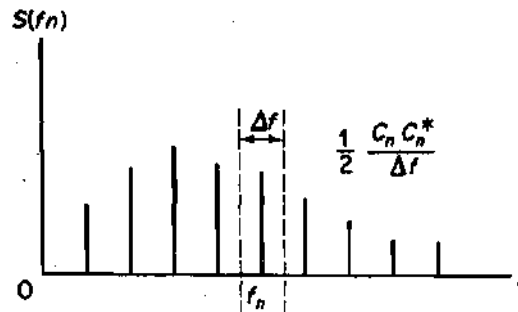


FIGURE 13.6.1. Discrete spectrum.

together and they more nearly resemble a continuous spectrum, as shown in Fig. 13.6.2. We now define the PSD, $S(f)$, for a continuous spectrum as the limiting case of $S(f_n)$ as $\Delta f \rightarrow 0$.

$$\lim_{\Delta f \rightarrow 0} S(f_n) = S(f) \quad (13.6.5)$$

The mean square value is then

$$\overline{x^2} = \int_0^{\infty} S(f) df \quad (13.6.6)$$

To illustrate the meaning of PS and PSD, the following experiment is described. A Xtal accelerometer is attached to a shaker, and its output is amplified, filtered, and read by a rms voltmeter, as shown by the block diagram of Fig. 13.6.3. The rms voltmeter should have a long time constant, which corresponds to a long averaging time.

We excite the shaker by a wide-band random input that is constant over the frequency range 0 to 2000 Hz. If the filter is bypassed, the rms voltmeter will read the rms vibration in the entire frequency spectrum. By assuming an ideal filter that will pass all vibrations of frequencies within the passband, the output of the filter represents a narrow-band vibration.

We consider a central frequency of 500 Hz and first set the upper and lower cut-off frequencies at 580 and 420 Hz, respectively. The rms meter will now read only the vibration within this 160-Hz band. Let us say that the reading is 8g. The mean square value is then $G(f_n) = 64g^2$, and its spectral density is $S(f_n) = 64g^2/160 = 0.40g^2/\text{Hz}$.

We next reduce the passband to 40 Hz by setting the upper and lower filter frequencies to 520 and 480 Hz, respectively. The mean square value passed by the filter is now one-quarter of the previous value, or $16g^2$, and the rms meter reads 4g.

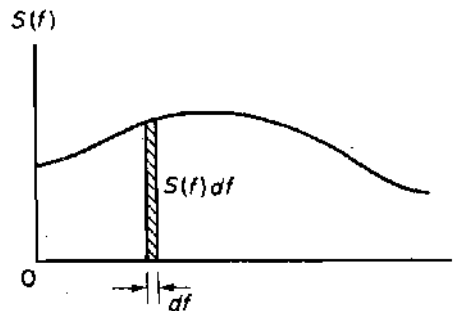


FIGURE 13.6.2. Continuous spectrum.

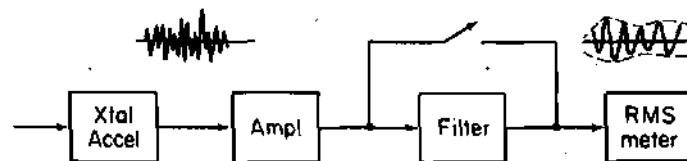


FIGURE 13.6.3. Measurement of random data.

By reducing the passband further to 10 Hz, between 505 Hz and 495 Hz, the rms meter reading becomes $2g$, as shown in the following tabulation:

Frequencies	Band-width	RMS Meter Reading	Filtered Mean Square	Spectral Density
f	Δf	$\sqrt{\overline{\delta(x^2)}}$	$G(f_n) = \Delta \overline{x^2}$	$S(f_n) = \frac{\Delta(\overline{x^2})}{\Delta f}$
580–420	160	$8g$	$64g^2$	$0.40g^2/\text{Hz}$
520–480	40	$4g$	$16g^2$	$0.40g^2/\text{Hz}$
505–495	10	$2g$	$4g^2$	$0.40g^2/\text{Hz}$

Note that as the bandwidth is reduced, the mean square value passed by the filter, or $G(f_n)$, is reduced proportionally. However, by dividing by the bandwidth, the density of the mean square value, $S(f_n)$, remains constant. The example clearly points out the advantage of plotting $S(f_n)$ instead of $G(f_n)$.

The PSD can also be expressed in terms of the delta function. As seen from Fig. 13.6.4, the area of a rectangular pulse of height $1/\Delta f$ and width Δf is always unity, and in the limiting case, when $\Delta f \rightarrow 0$, it becomes a delta function. Thus, $S(f)$ becomes

$$S(f) = \lim_{\Delta f \rightarrow 0} S(f_n) = \lim_{\Delta f \rightarrow 0} \frac{G(f_n)}{\Delta f} = G(f) \delta(f - f_n)$$

Typical spectral density functions for two common types of random records are shown in Figs. 13.6.5 and 13.6.6. The first is a wide-band noise-type of record that has a

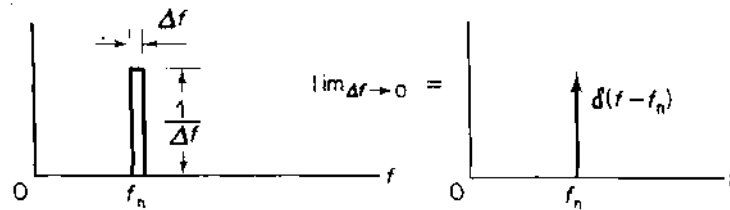


FIGURE 13.6.4. $\lim_{\Delta f \rightarrow 0} \frac{1}{\Delta f} = \delta(f - f_n)$.

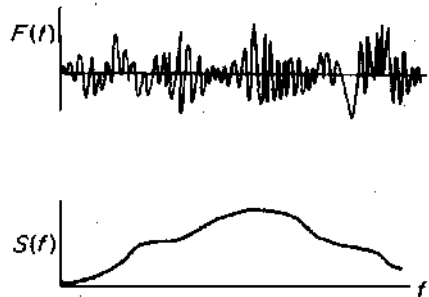


FIGURE 13.6.5. Wide-band record and its spectral density.

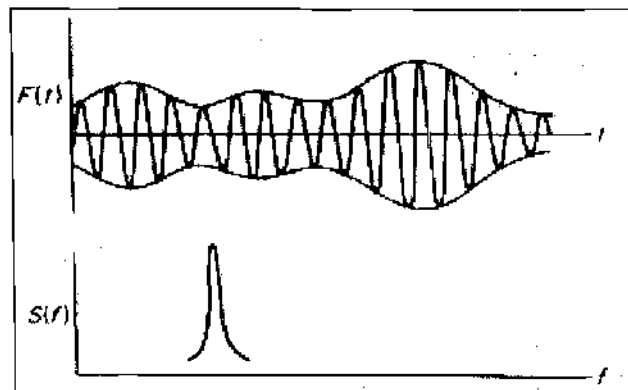


FIGURE 13.6.6. Narrow-band record and its spectral density.

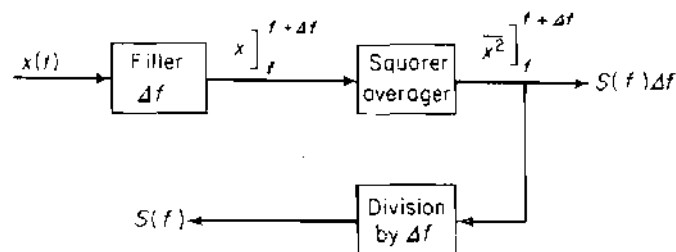


FIGURE 13.6.7. Power spectral density analyzer.

broad spectral density function. The second is a narrow-band random record that is typical of a response of a sharply resonant system to a wide-band input. Its spectral density function is concentrated around the frequency of the instantaneous variation within the envelope.

The spectral density of a given record can be measured electronically by the circuit of Fig. 13.6.7. Here the spectral density is noted as the contribution of the mean square value in the frequency interval Δf which is divided by Δf .

$$S(f) = \lim_{\Delta f \rightarrow 0} \frac{\Delta(\bar{x}^2)}{\Delta f} \quad (13.6.7)$$

The band-pass filter of passband $B = \Delta f$ passes $x(t)$ in the frequency interval f to $f + \Delta f$, and the output is squared, averaged, and divided by Δf .

For high resolution, Δf should be made as narrow as possible; however, the passband of the filter cannot be reduced indefinitely without losing the reliability of the measurement. Also, a long record is required for the true estimate of the mean square value, but actual records are always of finite length. It is evident now that a parameter of importance is the product of the record length and the bandwidth, $2BT$, which must sufficiently large.²

²See J. S. Bendat, and A. G. Piersol, *Random Data* (New York: John Wiley & Sons, 1971), p. 96.

EXAMPLE 13.6.1

A random signal has a spectral density that is a constant

$$S(f) = 0.004 \text{ cm}^2/\text{cps}$$

between 20 and 1200 cps and that is zero outside this frequency range. Its mean value is 2.0 cm. Determine its rms value and its standard deviation.

Solution The mean square value is found from

$$\overline{x^2} = \int_0^\infty S(f) df = \int_{20}^{1200} 0.004 df = 4.72$$

and the rms value is

$$\text{rms} = \sqrt{\overline{x^2}} = \sqrt{4.72} = 2.17 \text{ cm}$$

The variance σ^2 is defined by Eq. (13.2.6):

$$\begin{aligned}\sigma^2 &= \overline{x^2} - (\bar{x})^2 \\ &= 4.72 - 2^2 = 0.72\end{aligned}$$

and the standard deviation becomes

$$\sigma = \sqrt{0.72} = 0.85 \text{ cm}$$

The problem is graphically displayed by Fig. 13.6.8, which shows the time variation of the signal and its probability distribution.

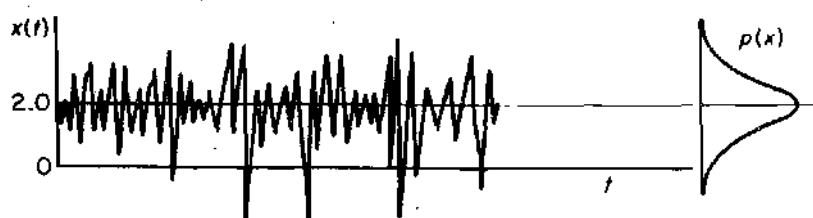


FIGURE 13.6.8.

EXAMPLE 13.6.2

Determine the Fourier coefficients C_n and the power spectral density of the periodic function shown in Fig. 13.6.9.

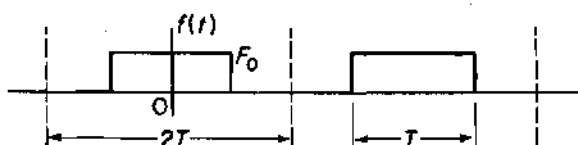


FIGURE 13.6.9.

Solution The period is $2T$ and C_n are

$$C_0 = \frac{2}{2T} \int_{-T/2}^{T/2} F_0 d\xi = F_0$$

$$C_n = \frac{2}{2T} \int_{-T/2}^{T/2} F_0 e^{-in\omega_0 \xi} d\xi = F_0 \left[\frac{\sin(n\pi/2)}{n\pi/2} \right]$$

Numerical values of C_n are computed as in the following table and plotted in Fig. 13.6.10.

n	$\frac{n\pi}{2}$	$\sin \frac{n\pi}{2}$	$\frac{1}{2} C_n$
0	0	0	$\frac{F_0}{2} = 1.0 \frac{F_0}{2}$
1	$\frac{\pi}{2}$	1	$\left(\frac{2}{\pi}\right) \frac{F_0}{2} = 0.636 \frac{F_0}{2}$
2	π	0	0
3	$\frac{3\pi}{2}$	-1	$\left(-\frac{2}{3\pi}\right) \frac{F_0}{2} = 0.212 \frac{F_0}{2}$
4	2π	0	0
5	$\frac{5\pi}{2}$	1	$\left(\frac{2}{5\pi}\right) \frac{F_0}{2} = 0.127 \frac{F_0}{2}$

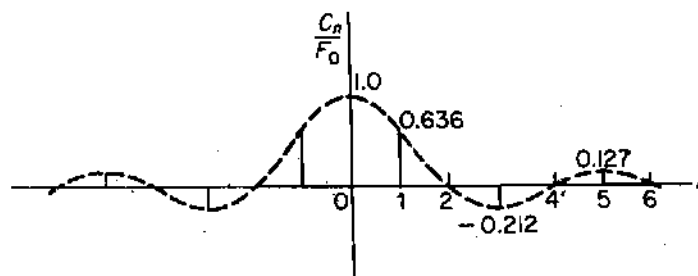


FIGURE 13.6.10. Fourier coefficients versus n .

The mean square value is determined from the equation

$$\begin{aligned} \overline{x^2} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{1}{4} \left[\sum_n (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t}) \right]^2 dt \\ &= \sum_{n=1}^{\infty} \frac{C_n C_n^*}{2} \end{aligned}$$

and because $\overline{x^2} = \int_{-\infty}^{\infty} S_f(\omega) d\omega$, the spectral density function can be represented by a series of delta functions:

$$S_f(\omega) = \sum_{n=1}^{\infty} \frac{C_n C_n^*}{2} \delta(\omega - n\omega_0)$$

3.7 FOURIER TRANSFORMS

The discrete frequency spectrum of periodic functions becomes a continuous one when the period T is extended to infinity. Random vibrations are generally not periodic and the determination of its continuous frequency spectrum requires the use of the Fourier integral, which can be regarded as a limiting case of the Fourier series as the period approaches infinity.

The Fourier transform has become the underlying operation for the modern time series analysis. In many of the modern instruments for spectral analysis, the calculation performed is that of determining the amplitude and phase of a given record.

The *Fourier integral* is defined by the equation

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df \quad (13.7.1)$$

In contrast to the summation of the discrete spectrum of sinusoids in the Fourier series, the Fourier integral can be regarded as a summation of the continuous spectrum of sinusoids. The quantity $X(f)$ in the previous equation is called the *Fourier transform* of $x(t)$, which can be evaluated from the equation

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt \quad (13.7.2)$$

Like the Fourier coefficient C_n , $X(f)$ is a complex quantity which is a continuous function of f from $-\infty$ to $+\infty$. Equation (13.7.2) resolves the function $x(t)$ into harmonic components $X(f)$ whereas Eq. (13.7.1) synthesizes these harmonic components to the original time function $x(t)$. The two previous equations above are referred to as the *Fourier transform pair*.

Fourier transform (FT) of basic functions. To demonstrate the spectral character of the FT, we consider the FT of some basic functions.

EXAMPLE 13.7.1

$$x(t) = A e^{i2\pi f_n t} \quad (a)$$

From Eq. (13.7.1), we have

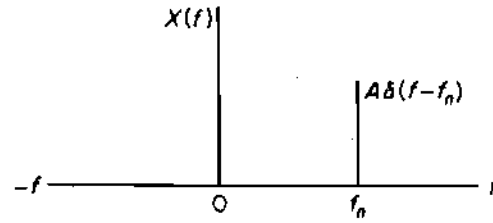
$$A e^{i2\pi f_n t} = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df$$

Recognizing the properties of a delta function, this equation is satisfied if

$$X(f) = A \delta(f - f_n) \quad (b)$$

Substituting into Eq. (13.7.2), we obtain

$$\delta(f - f_n) = \int_{-\infty}^{\infty} e^{-i2\pi(f-f_n)t} dt$$

FIGURE 13.7.1. FT of $Ae^{i2\pi f_n t}$.

The FT of $x(t)$ is displayed in Fig. 13.7.1, which demonstrates its spectral character.

EXAMPLE 13.7.2

$$x(t) = a_n \cos(2\pi f_n t) \quad (a)$$

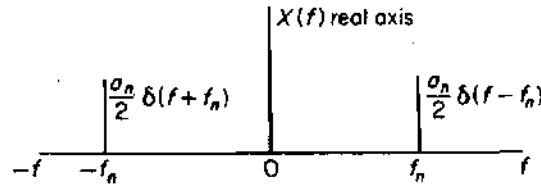
Because

$$\cos 2\pi f_n t = \frac{1}{2}(e^{i2\pi f_n t} + e^{-i2\pi f_n t})$$

the result of Example 13.7.1 immediately gives

$$X(f) = \frac{a_n}{2} [\delta(f - f_n) + \delta(f + f_n)] \quad (b)$$

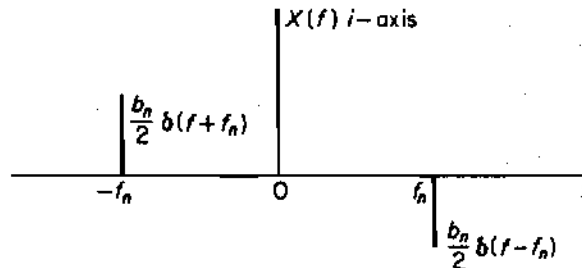
Figure 13.7.2 shows that $X(f)$ is a two-sided function of f .

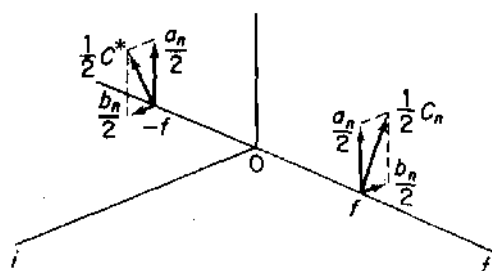
FIGURE 13.7.2. FT of $a_n \cos 2\pi f_n t$.

In a similar manner, the FT of $b_n \sin 2\pi f_n t$ is

$$X(f) = -i \frac{b_n}{2} [\delta(f - f_n) - \delta(f + f_n)]$$

which is shown on the imaginary plane of Fig. 13.7.3.

FIGURE 13.7.3. FT of $b_n \sin 2\pi f_n t$.

FIGURE 13.7.4 FT of $a_n \cos 2\pi f_n t + b_n \sin 2\pi f_n t$.

If we put the two FTs together in perpendicular planes, as shown in Fig. 13.7.4, we obtain the complex conjugate coefficients $C_n = a_n - ib_n$ and $C_n^* = a_n + ib_n$. Thus, the product

$$\frac{C_n C_n^*}{4} = \frac{1}{4} (a_n^2 + b_n^2) = c_n c_n^*$$

is the square of the magnitude of the Fourier series, which is generally plotted at $\pm f$.

EXAMPLE 13.7.3

We next determine the FT of a rectangular pulse, which is an example of an aperiodic function. (See Fig. 13.7.5.) Its FT is

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt = \int_{-T/2}^{T/2} A e^{-i2\pi ft} dt = AT \left(\frac{\sin \pi f T}{\pi f T} \right)$$

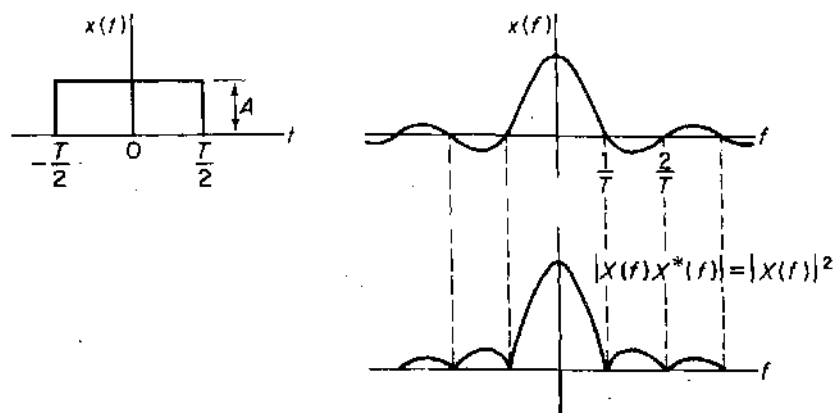


FIGURE 13.7.5. Rectangular pulse and its spectra.

Note that the FT is now a continuous function instead of a discontinuous function. The product XX^* , which is a real number, is also plotted here. Later it will be shown to be equal to the spectral density function.

FTs of derivatives. When the FT is expressed in terms of ω instead of f , a factor $1/2\pi$ is introduced in the equation for $x(t)$:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \quad (13.7.3)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \quad (13.7.4)$$

This form is sometimes preferred in developing mathematical relationships. For example, if we differentiate Eq. (13.7.3) with respect to t , we obtain the FT pair:

$$\begin{aligned} \dot{x}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [i\omega X(\omega)] e^{i\omega t} d\omega \\ i\omega X(\omega) &= \int_{-\infty}^{\infty} \dot{x}(t) e^{-i\omega t} dt \end{aligned}$$

Thus, the FT of a derivative is simply the FT of the function multiplied by $i\omega$:

$$FT[\dot{x}(t)] = i\omega FT[x(t)] \quad (13.7.5)$$

Differentiating again, we obtain

$$FT[\ddot{x}(t)] = -\omega^2 FT[x(t)] \quad (13.7.6)$$

These equations enable one to conveniently take the FT of differential equations. For example, if we take the FT of the differential equation

$$m\ddot{y} + c\dot{y} + ky = x(t)$$

we obtain

$$(-m\omega^2 + i\omega c + k)Y(\omega) = X(\omega)$$

where $X(\omega)$ and $Y(\omega)$ are the FT of $x(t)$ and $Y(t)$, respectively.

Parseval's theorem. Parseval's theorem is a useful tool for converting time integration into frequency integration. If $X_1(f)$ and $X_2(f)$ are Fourier transforms of real time functions $x_1(t)$ and $x_2(t)$, respectively, Parseval's theorem states that

$$\begin{aligned} \int_{-\infty}^{\infty} x_1(t)x_2(t) dt &= \int_{-\infty}^{\infty} X_1(f)X_2^*(f) df \\ &= \int_{-\infty}^{\infty} X_1^*(f)X_2(f) df \end{aligned} \quad (13.7.7)$$

This relationship may be proved using the Fourier transform as follows:

$$\begin{aligned} x_1(t)x_2(t) &= x_2(t) \int_{-\infty}^{\infty} X_1(f) e^{i2\pi ft} df \\ \int_{-\infty}^{\infty} x_1(t)x_2(t) dt &= \int_{-\infty}^{\infty} x_2(t) \int_{-\infty}^{\infty} X_1(f) e^{i2\pi ft} df dt \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} X_1(f) \left[\int_{-\infty}^{\infty} x_2(t) e^{i2\pi ft} dt \right] df \\
&= \int_{-\infty}^{\infty} X_1(f) X_2^*(f) df
\end{aligned}$$

All the previous formulas for the mean square value, autocorrelation, and cross correlation can now be expressed in terms of the Fourier transform by Parseval's theorem.

EXAMPLE 13.7.4

Express the mean square value in terms of the Fourier transform. Letting $x_1(t) = x_2(t) = x(t)$, and averaging over T , which is allowed to go to ∞ , we obtain

$$\overline{x^2} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} X(f) X^*(f) df$$

Comparing this with Eq. (13.6.6), we obtain the relationship

$$S(f_{\pm}) = \lim_{T \rightarrow \infty} \frac{1}{T} X(f) X^*(f) \quad (13.7.8)$$

where $S(f_{\pm})$ is the spectral density function over positive and negative frequencies.

EXAMPLE 13.7.5

Express the autocorrelation in terms of the Fourier transform. We begin with the Fourier transform of $x(t + \tau)$:

$$x(t + \tau) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f(t+\tau)} df$$

Substituting this into the expression for the autocorrelation, we obtain

$$\begin{aligned}
R(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} x(t) x(t + \tau) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} e^{i2\pi f\tau} df dt \\
&= \int_{-\infty}^{\infty} \frac{1}{T} \left[\int_{-\infty}^{\infty} x(t) e^{i2\pi ft} dt \right] X(f) e^{i2\pi f\tau} df \\
&= \int_{-\infty}^{\infty} \left[\lim_{T \rightarrow \infty} \frac{1}{T} X^*(f) X(f) \right] e^{i2\pi f\tau} df
\end{aligned}$$

By substituting from Eq. (13.7.8) the preceding equation becomes

$$R(\tau) = \int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau} df \quad (13.7.9)$$

The inverse of the preceding equation is also available from the Fourier transform:

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau \quad (13.7.10)$$

Because $R(\tau)$ is symmetric about $\tau = 0$, the last equation can also be written as

$$S(f) = 2 \int_0^{\infty} R(\tau) \cos 2\pi f\tau d\tau \quad (13.7.11)$$

These are the *Wiener-Khintchine* equations, and they state that the spectral density function is the FT of the autocorrelation function.

As a parallel to the Wiener-Khintchine equations, we can define the cross correlation between two quantities $x(t)$ and $y(t)$ as

$$\begin{aligned} R_{xy}(\tau) &= \langle x(t)y(t+\tau) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t+\tau) dt \\ &= \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} X^*(f)Y(f) e^{i2\pi f\tau} df \\ R_{xy}(\tau) &= \int_{-\infty}^{\infty} S_{xy}(f) e^{i2\pi f\tau} df \end{aligned} \quad (13.7.12)$$

where the cross-spectral density is defined as

$$\begin{aligned} S_{xy}(f) &= \lim_{T \rightarrow \infty} \frac{1}{T} X^*(f)Y(f) \quad -\infty \leq f \leq \infty \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} X(f)Y^*(f) \\ &= S_{xy}^*(f) = S_{xy}(-f) \end{aligned} \quad (13.7.13)$$

Its inverse from the Fourier transform is

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i2\pi f\tau} d\tau \quad (13.7.14)$$

which is parallel to Eq. (13.7.10). Unlike the autocorrelation, the cross-correlation and the cross-spectral density functions are, in general, not even functions; hence, the limits $-\infty$ to $+\infty$ are retained.

EXAMPLE 13.7.6

Using the relationship

$$S(f) = 2 \int_0^{\infty} R(\tau) \cos 2\pi f\tau d\tau$$

and the results of Example 13.5.1,

$$R(\tau) = A^2(T - |\tau|)$$

find $S(f)$ for the rectangular pulse.

Solution Because $R(\tau) = 0$ for τ outside $\pm T$, we have

$$\begin{aligned}
 S(f) &= 2 \int_0^T A^2(T - \tau) \cos 2\pi f\tau \, d\tau \\
 &= 2A^2T \int_0^T \cos 2\pi f\tau \, d\tau - 2A^2 \int_0^T \tau \cos 2\pi f\tau \, d\tau \\
 &= 2A^2T \left. \frac{\sin 2\pi f\tau}{2\pi f} \right|_0^T - 2A^2 \left[\frac{\cos 2\pi f\tau}{(2\pi f)^2} + \frac{\tau}{2\pi f} \sin 2\pi f\tau \right]_0^T \\
 &= \frac{2A^2}{(2\pi f)^2} (1 - \cos 2\pi fT) = A^2T^2 \left(\frac{\sin \pi fT}{\pi fT} \right)^2
 \end{aligned}$$

Thus, the power spectral density of a rectangular pulse using Eq. (13.7.11) is

$$S(f) = A^2T^2 \left(\frac{\sin \pi fT}{\pi fT} \right)^2$$

Note from Example 13.7.3 that this is also equal to $X(f)X^*(f) = |X(f)|^2$.

EXAMPLE 13.7.7

Show that the frequency response function $H(\omega)$ is the Fourier transform of the impulse response function $h(t)$.

Solution From the convolution integral, Eq. (4.2.1), the response equation in terms of the impulse response function is

$$x(t) = \int_{-\infty}^t f(\xi)h(t - \xi) \, d\xi$$

where the lower limit has been extended to $-\infty$ to account for all past excitations. By letting $\tau = (t - \xi)$, the last integral becomes

$$x(t) = \int_0^\infty f(t - \tau)h(\tau) \, d\tau$$

For a harmonic excitation $f(t) = e^{i\omega t}$, the preceding equation becomes

$$\begin{aligned}
 x(t) &= \int_0^\infty e^{i\omega(t-\tau)}h(\tau) \, d\tau \\
 &= e^{i\omega t} \int_0^\infty h(\tau)e^{-i\omega\tau} \, d\tau
 \end{aligned}$$

Because the steady-state output for the input $y(t) = e^{i\omega t}$ is $x = H(\omega)e^{i\omega t}$, the frequency-response function is

$$H(\omega) = \int_0^\infty h(\tau)e^{-i\omega\tau} \, d\tau = \int_{-\infty}^\infty h(\tau)e^{-i\omega\tau} \, d\tau$$

which is the FT of the impulse response function $h(t)$. The lower limit in the preceding integral has been changed from 0 to $-\infty$ because $h(t) = 0$ for negative t .

13.8 FTS AND RESPONSE

In engineering design, we often need to know the relationship between different points in the system. For example, how much of the roughness of a typical road is transmitted through the suspension system to the body of an automobile? (Here the term transfer function³ is often used for the frequency-response function.) Furthermore, it is often not possible to introduce a harmonic excitation to the input point of the system. It may be necessary to accept measurements $x(t)$ and $y(t)$ at two different points in the system for which the frequency response function is desired. The frequency response function for these points can be obtained by taking the FT of the input and output. The quantity $H(\omega)$ is then available from

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\text{FT of output}}{\text{FT of input}} \quad (13.8.1)$$

where $X(\omega)$ and $Y(\omega)$ are the FT of $x(t)$ and $y(t)$.

If we multiply and divide this equation by the complex conjugate $X^*(\omega)$, the result is

$$H(\omega) = \frac{Y(\omega)X^*(\omega)}{X(\omega)X^*(\omega)} \quad (13.8.2)$$

The denominator $X(\omega)X^*(\omega)$ is now a real quantity. The numerator is the cross spectrum $Y(\omega)X^*(\omega)$ between the input and the output and is a complex quantity. The phase of $H(\omega)$ is then found from the real and imaginary parts of the cross spectrum, which is simply

$$|Y(\omega)|/\phi_y \cdot |X^*(\omega)|/\phi_x = |Y(\omega)X^*(\omega)|/\phi_y - \phi_x$$

Another useful relationship can be found by multiplying $H(\omega)$ by its conjugate $H^*(\omega)$. The result is

$$H(\omega)H^*(\omega) = \frac{Y(\omega)Y^*(\omega)}{X(\omega)X^*(\omega)}$$

or

$$Y(\omega)Y^*(\omega) = |H(\omega)|^2 X(\omega)X^*(\omega) \quad (13.8.4)$$

Thus, the output power spectrum is equal to the square of the system transfer function multiplied by the input power spectrum. Obviously, each side of the previous equation is real and the phase does not enter in.

We wish now to examine the mean square value of the response. From Eq. (13.7.8), the mean square value of the input $x(t)$ is

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(f) df = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} X(f)X^*(f) df$$

³Strictly speaking, the transfer function is the ratio of the Laplace transform of the output to the Laplace transform of the input. In the frequency domain, however, the real part of $s = \alpha + i\omega$ is zero, and the LT becomes the FT.

The mean square value of the output $y(t)$ is

$$\overline{y^2} = \int_{-\infty}^{\infty} S_y(f_{\pm}) df = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} Y(f) Y^*(f) df$$

Substituting $YY^* = |H(f)|^2 XX^*$, we obtain

$$\begin{aligned} \overline{y^2} &= \int_{-\infty}^{\infty} |H(f)|^2 \left[\lim_{T \rightarrow \infty} \frac{1}{T} X(f) X^*(f) \right] df \\ &= \int_{-\infty}^{\infty} |H(f)|^2 S_x(f_{\pm}) df \end{aligned} \quad (13.8.5)$$

which is the mean square value of the response in terms of the system response function and the spectral density of the input.

In these expressions, $S(f_{\pm})$ are the two-sided spectral density functions over both the positive and negative frequencies. Also, $S(f_{\pm})$ are even functions. In actual practice, it is desirable to work with spectral densities over only the positive frequencies. Equation (13.8.5) can then be written as

$$\overline{y^2} = \int_0^{\infty} |H(f)|^2 S_x(f_{+}) df \quad (13.8.6)$$

and because the two expressions must result in the same value for the mean square value, the relationship between the two must be

$$S(f_{+}) = S(f) = 2S(f_{\pm}) \quad (13.8.7)$$

Some authors also use the expression

$$\overline{y^2} = \int_0^{\infty} |H(\omega)|^2 S_x(\omega) d\omega \quad (13.8.8)$$

Again, the equations must result in the same mean square value so that

$$2\pi S(\omega) = S(f) \quad (13.8.9)$$

For a single-DOF system, we have

$$H(f) = \frac{1/k}{[1 - (f/f_n)^2] + i[2\zeta(f/f_n)]} \quad (13.8.10)$$

If the system is lightly damped, the response function $H(f)$ is peaked steeply at resonance, and the system acts like a narrow-band filter. If the spectral density of excitation is broad, as in Fig. 13.8.1, the mean square response for the single-DOF system can be approximated by the equation

$$\overline{y^2} \cong \frac{f_n}{k^2} S_x(f_n) \frac{\pi}{4\zeta} \quad (13.8.11)$$

where

$$\frac{\pi}{4\zeta} = \int_0^{\infty} \frac{d(f/f_n)}{[1 - (f/f_n)^2]^2 + [2\zeta(f/f_n)]^2}$$

and $S_x(f_n)$ is the spectral density of the excitation at frequency f_n .

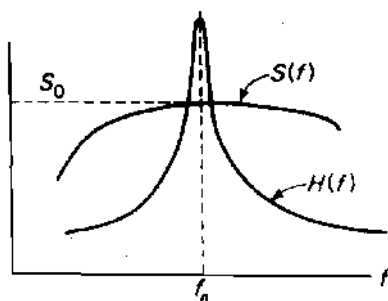


FIGURE 13.8.1. $S(f)$ and $H(f)$ leading to $\bar{y}^2$ of Equation 13.7.9.

EXAMPLE 13.8.1

The response of any structure to a single-point random excitation can be computed by a simple numerical procedure, provided the spectral density of the excitation and the frequency response curve of the structure are known. For example, consider the structure of Fig. 13.8.2(a), whose base is subjected to a random acceleration input with the power spectral density function shown in Fig. 13.8.2(b). It is desired to compute the response of the point p and establish the probability of exceeding any specified acceleration.

The frequency response function $H(f)$ for the point p can be obtained experimentally by applying to the base a variable frequency sinusoidal shaker with a constant acceleration input a_0 , and measuring the acceleration response at p . Dividing the measured acceleration by a_0 , $H(f)$ may appear as in Fig. 13.8.2(c).

The mean square response $\bar{a}_p^2$ at p is calculated numerically from the equation

$$\bar{a}_p^2 = \sum_i S(f_i) |H(f_i)|^2 \Delta f_i$$

Table 13.8.1 illustrates the computational procedure.

The probability of exceeding specified accelerations are

$$p[|a| > 26.6g] = 31.7\%$$

$$p[a_{\text{peak}} > 26.6g] = 60.7\%$$

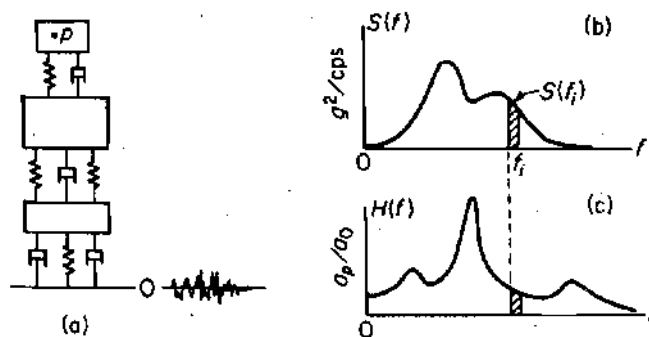


FIGURE 13.8.2.

Table 13.8.1 NUMERICAL EXAMPLE

f (cps)	Δf (cps)	$S(f)$ (g^2/cps)	$ H(F_i) $ (Nondimensional)	$ H(f_i) ^2 \Delta f$ (cps)	$S(f_i) H(f_i) ^2 \Delta f$ (g^2 units)
0	10	0	1.0	10.	0
10	10	0	1.0	10.	0
20	10	0.2	1.1	12.1	2.4
30	10	0.6	1.4	19.6	11.8
40	10	1.2	2.0	40.	48.0
50	10	1.8	1.3	16.9	30.5
60	10	1.8	1.3	16.9	30.5
70	10	1.1	2.0	40.	44.0
80	10	0.9	3.7	137.	123.
90	10	1.1	5.4	291.	320.
100	10	1.2	2.2	48.4	57.7
110	10	1.1	1.3	16.9	18.6
120	10	0.8	0.8	6.4	5.1
130	10	0.6	0.6	3.6	2.2
140	10	0.3	0.5	2.5	0.8
150	10	0.2	0.6	3.6	0.7
160	10	0.2	0.7	4.9	0.1
170	10	0.1	1.3	16.9	1.7
180	10	0.1	1.1	12.1	1.2
190	10	0.5	0.7	4.9	2.3
200	10	0	0.5	2.5	0
210	10	0	0.4	1.6	0

$\bar{a}^2 = 700.6g^2$
 $\sigma = \sqrt{700.6g^2} = 26.6g$

$$p[|a| > 79.8g] = 0.3\%$$

$$p[a_{\text{peak}} > 79.8g] = 1.2\%$$

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PROBLEMS

- 13.1 Give examples of random data and indicate classifications for each example.
 13.2 Discuss the differences between nonstationary, stationary, and ergodic data.
 13.3 Discuss what we mean by the expected value. What is the expected number of heads when eight coins are thrown 100 times; 1000 times? What is the probability for tails?
 13.4 Throw a coin 50 times, recording 1 for head and 0 for tail. Determine the probability of obtaining heads by dividing the cumulative heads by the number of throws and plot this number as a function of the number of throws. The curve should approach 0.5.
 13.5 For the series of triangular waves shown in Fig. P13.5, determine the mean and the mean square values.

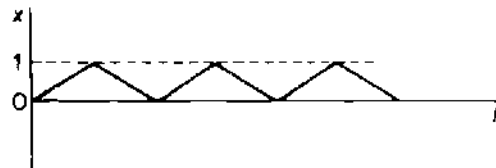


FIGURE P13.5.

- 13.6 A sine wave with a steady component has the equation

$$x = A_0 + A_1 \sin \omega t$$

Determine the expected values $E(x)$ and $E(x^2)$.

- 13.7 Determine the mean and mean square values for the rectified sine wave.
 13.8 Discuss why the probability distribution of the peak values of a random function should follow the Rayleigh distribution or one similar in shape to it.
 13.9 Show that for the Gaussian probability distribution $p(x)$, the central moments are given by

$$E(x^n) = \int_{-\infty}^{\infty} x^n p(x) dx$$

$$= \begin{cases} 0 & \text{for } n \text{ odd} \\ 1 \cdot 3 \cdot 5 \cdots (n-1) \sigma^n & \text{for } n \text{ even} \end{cases}$$

- 13.10 Derive the equations for the cumulative probability and the probability density functions of the sine wave. Plot these results.
 13.11 What would the cumulative probability and the probability density curves look like for the rectangular wave shown in Fig. P13.11?

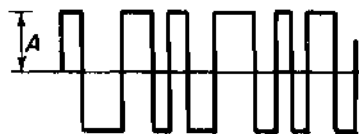


FIGURE P13.11.

13.12 Determine the autocorrelation of a cosine wave $x(t) = A \cos t$, and plot it against τ .

13.13 Determine the autocorrelation of the rectangular wave shown in Fig. P13.13.

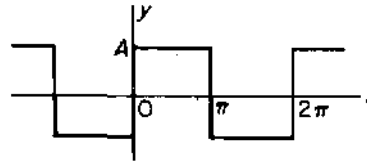


FIGURE P13.13.

13.14 Determine the autocorrelation of the rectangular pulse and plot it against τ .

13.15 Determine the autocorrelation of the binary sequence shown in Fig. P13.15. *Suggestion:* Trace the wave on transparent graph paper and shift it through τ .



FIGURE P13.15.

13.16 Determine the autocorrelation of the triangular wave shown in Fig. P13.16.

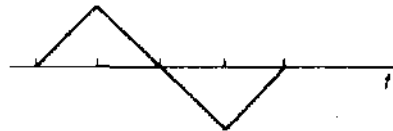


FIGURE P13.16.

13.17 Figure P13.17 shows the acceleration spectral density plot of a random vibration. Approximate the area by a rectangle and determine the rms value in m/s^2 .

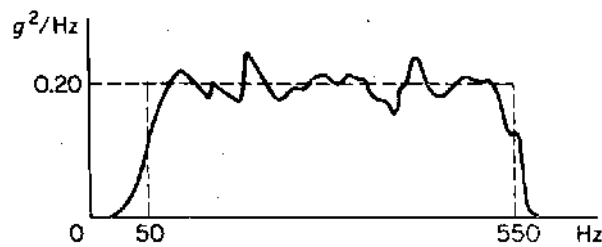


FIGURE P13.17.

13.18 Determine the rms value of the spectral density plot shown in Fig. P13.18.

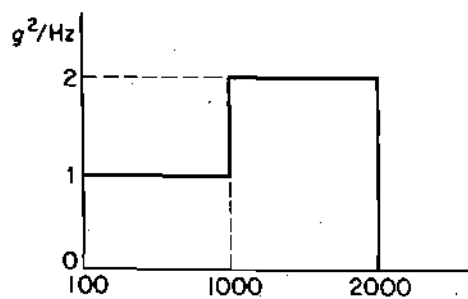


FIGURE P13.18.

- 13.19** The power spectral density plot of a random vibration is shown in Fig. P13.19. The slopes represent a 6-dB/octave. Replot the result on a linear scale and estimate the rms value.

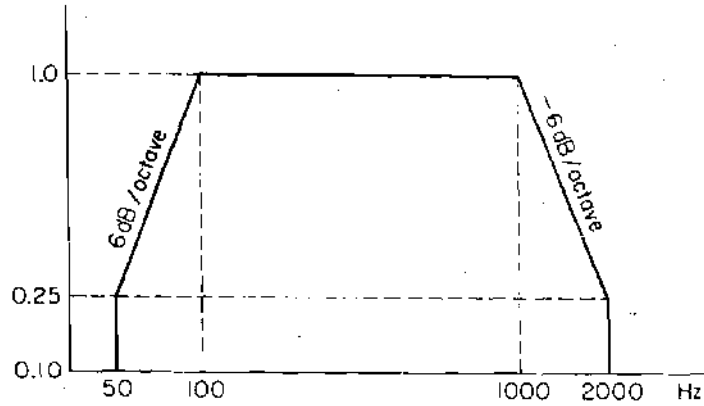


FIGURE P13.19.

- 13.20** Determine the spectral density function for the waves in Fig. P13.20.

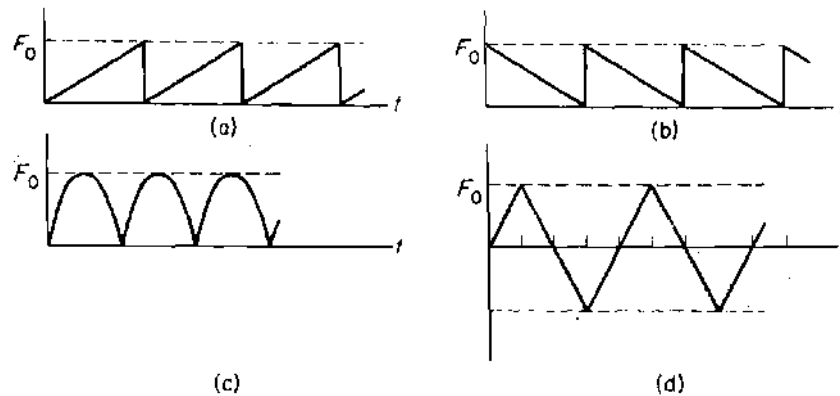


FIGURE P13.20.

- 13.21** A random signal is found to have a constant spectral density of $S(f) = 0.002 \text{ in.}^2/\text{cps}$ between 20 and 2000 cps. Outside this range, the spectral density is zero. Determine the standard deviation and the rms value if the mean value is 1.732 in. Plot this result.

- 13.22** Derive the equation for the coefficients C_n of the periodic function

$$f(t) = \text{Re} \sum_{n=0}^{\infty} C_n e^{in\omega_0 t}$$

- 13.23** Show that for Prob. 13.22, $C_{-n} = C_n^*$, and that $f(t)$ can be written as

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

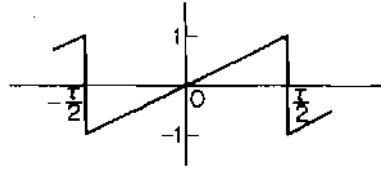


FIGURE P13.24.

- 13.24** Determine the Fourier series for the sawtooth wave shown in Fig. P13.24 and plot its spectral density.

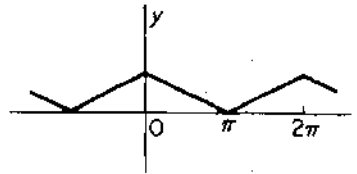


FIGURE P13.25.

- 13.25** Determine the complex form of the Fourier series for the wave shown in Fig. P13.25 and plot its spectral density.
- 13.26** Determine the complex form of the Fourier series for the rectangular wave shown in Fig. P13.13 and plot its spectral density.
- 13.27** The sharpness of the frequency-response curve near resonance is often expressed in terms of $Q = \frac{1}{2}\zeta$. Points on either side of resonance where the response falls to a value $1/\sqrt{2}$ are called half-power points. Determine the respective frequencies of the half-power points in terms of ω_n and Q .
- 13.28** Show that

$$\int_0^\infty \frac{d\eta}{(1 - \eta^2)^2 + (2\zeta\eta)^2} = \frac{\pi}{4\zeta} \quad \text{for } \zeta \ll 1$$

- 13.29** The differential equation of a system with structural damping is given as

$$m\ddot{x} + k(1 + i\gamma)x = F(t)$$

Determine the frequency-response function.

- 13.30** A single-DOF system with natural frequency ω_n and damping factor $\zeta = 0.10$ is excited by the force

$$F(t) = F \cos(0.5\omega_n t - \theta_1) + F \cos(\omega_n t - \theta^2) + F \cos(2\omega_n t - \theta_3)$$

Show that the mean square response is

$$\begin{aligned} \overline{y^2} &= (1.74 + 25.0 + 0.110) \frac{1}{2} \left(\frac{F}{k} \right)^2 \\ &= 13.43 \left(\frac{F}{k} \right)^2 \end{aligned}$$

- 13.31** In Example 13.7.3, what is the probability of the instantaneous acceleration exceeding a value $53.2g$? Of the peak value exceeding this value?
- 13.32** A large hydraulic press stamping out metal parts is operating under a series of forces approximated by Fig. P13.32. The mass of the press on its foundation is 40 kg and its

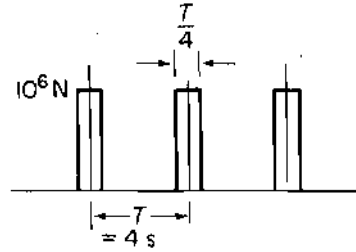


FIGURE P13.32.

natural frequency is 2.20 Hz. Determine the Fourier spectrum of the excitation and the mean square value of the response.

- 13.33** For a single-DOF system, the substitution of Eq. (13.8.10) into Eq. (13.8.6) results in

$$\overline{y^2} = \int_0^\infty S_x(f) \frac{1}{k^2} \frac{df}{[1 - (f/f_n)^2]^2 + [2\zeta(f/f_n)]^2}$$

where $S_x(f_+)$ is the spectral density of the excitation force. When the damping ζ is small and the variation of $S_x(f_+)$ is gradual, the last equation becomes

$$\begin{aligned} \overline{y^2} &\approx S_x(f_n) \frac{f_n}{k^2} \int_0^\infty \frac{d(f/f_n)}{[1 - (f/f_n)^2]^2 + [2\zeta(f/f_n)]^2} \\ &= S_x(f_n) \frac{f_n}{k^2} \frac{\pi}{4\zeta} \end{aligned}$$

which is Eq. (13.8.11). Derive a similar equation for the mean square value of the relative motion z of a single-DOF system excited by the base motion, in terms of the spectral density $S_y(f_+)$ of the base acceleration. (See Sec. 3.5.) If the spectral density of the base acceleration is constant over a given frequency range, what must be the expression for $\overline{z^2}$?

- M 13.34** Referring to Sec. 3.5, we can write the equation for the absolute acceleration of the mass undergoing base excitation as

$$\ddot{x} = \frac{k + i\omega c}{k - m\omega^2 + i\omega c} \cdot \ddot{y}$$

Determine the equation for the mean square acceleration $\overline{\ddot{x}^2}$. Establish a numerical integration technique for the computer evaluation of $\overline{\ddot{x}^2}$.

- 13.35** A radar dish with a mass of 60 kg is subject to wind loads with the spectral density shown in Fig. P13.35. The dish-support system has a natural frequency of 4 Hz. Determine the

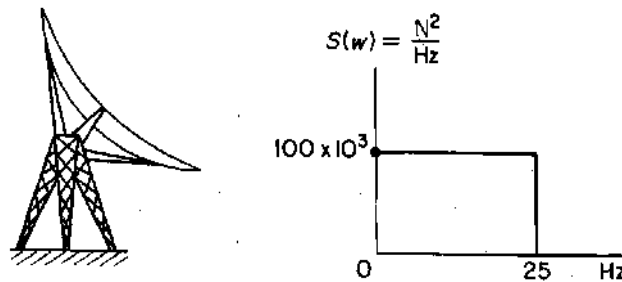


FIGURE P13.35.

mean square response and the probability of the dish exceeding a vibration amplitude of 0.132 m. Assume $\zeta = 0.05$.

- 13.36** A jet engine with a mass of 272 kg is tested on a stand, which results in a natural frequency of 26 Hz. The spectral density of the jet force under test is shown in Fig. P13.36. Determine the probability of the vibration amplitude in the axial direction of the jet thrust exceeding 0.012 m. Assume $\zeta = 0.10$.

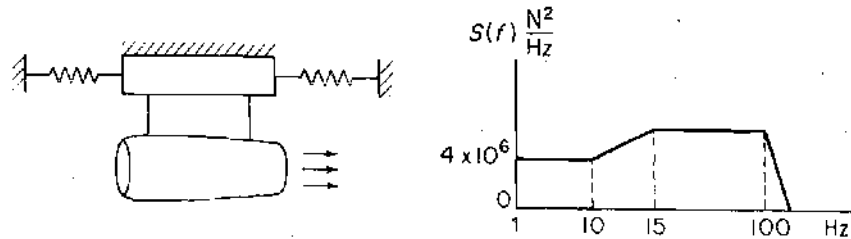


FIGURE P13.36.

- 13.37** An SDF system with viscous damping $\zeta = 0.03$ is excited by white-noise excitation $F(t)$ having a constant power spectral density of $5 \times 10^6 \text{ N}^2/\text{Hz}$. The system has a natural frequency of $\omega_n = 30 \text{ rad/s}$ and a mass of 1500 kg. Determine σ . Assuming Rayleigh distribution for peaks, determine the probability that the maximum peak response will exceed 0.037 m.

- 13.38** Starting with the relationship

$$x(t) = \int_0^\infty f(t - \xi)h(\xi) d\xi$$

and using the FT technique, show that

$$X(i\omega) = F(i\omega)H(i\omega)$$

and

$$\overline{x^2} = \int_0^\infty S_F(\omega)|H(i\omega)|^2 d\omega$$

where

$$S_F(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} F(i\omega)F^*(i\omega)$$

- 13.39** Starting with the relationship

$$H(i\omega) = |H(i\omega)|e^{i\phi(\omega)}$$

show that

$$\frac{H(i\omega)}{H^*(i\omega)} = e^{i2\phi(\omega)}$$

13.40 Find the frequency spectrum of the rectangular pulse shown in Fig. P13.40.

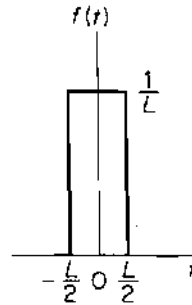


FIGURE P13.40.

13.41 Show that the unit step function has no Fourier transform. *Hint:* Examine

$$\int_{-\infty}^{\infty} |f(t)| dt$$

13.42 Starting with the equations

$$\begin{aligned} S_{FX}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} F^*(i\omega) X(i\omega) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} F(FH) = S_F H \end{aligned}$$

and

$$\begin{aligned} S_{XF}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} X^* F \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} (F^* H^*) F = S_F H^* \end{aligned}$$

show that

$$\frac{S_{FX}(\omega)}{S_{XF}(\omega)} = e^{i2\phi(\omega)}$$

and

$$\frac{S_F(\omega)}{S_{XF}(\omega)} = \frac{S_{FX}(\omega)}{S_F(\omega)} = H(i\omega)$$

13.43 The differential equation for the longitudinal motion of a uniform slender rod is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Show that for an arbitrary axial force at the end $x = 0$, with the other end $x = l$ free, the Laplace transform of the response is

$$\bar{u}(x,s) = \frac{-c\bar{F}(s)e^{-s(l/c)}}{sAE(1 - e^{-2s(l/c)})} [e^{(s/c)(x-l)} + e^{-(s/c)(s-l)}]$$

- 13.44** If the force in Prob. 13.43 is harmonic and equal to $F(t) = F_0 e^{i\omega t}$, show that

$$u(x,t) = \frac{cF_0 e^{i\omega t} \cos[(\omega l/c)(x/l - l)]}{\omega AE \sin(\omega l/c)}$$

and

$$\sigma(x,t) = \frac{-\sin[(\omega l/c)(x/l - 1)]}{\sin(\omega l/c)} \frac{F_0}{A} e^{i\omega t}$$

where σ is the stress.

- 13.45** With $S(\omega)$ as the spectral density of the excitation stress at $x = 0$, show that the mean square stress in Prob. 13.43 is

$$\overline{\sigma^2} \cong \frac{2\pi}{\gamma} \sum_n \frac{c}{n\pi l} S(\omega_n) \sin^2 n\pi \frac{x}{l}$$

where structural damping is assumed. The normal modes of the problem are

$$\varphi_n(x) = \sqrt{2} \cos n\pi \left(\frac{x}{l} - 1 \right),$$

$$\omega_n = n\pi \left(\frac{c}{l} \right), \quad c = \sqrt{\frac{AE}{m}}$$

- 13.46** Determine the FT of $x(t - t_0)$ and show that it is equal to $e^{-i2\pi f t_0} X(f)$, where $X(f) = \text{FT}[x(t)]$.

- 13.47** Prove that the FT of a convolution is the product of the separate FT.

$$\text{FT}[x(t)*y(t)] = X(f)Y(f)$$

- 13.48** Using the derivative theorem, show that the FT of the derivative of a rectangular pulse is a sine wave.

- 13.49** For some random variables the integral in Eq. (13.4.4) does not converge. Consider a function given by

$$p(x) = \frac{a}{b + x^2}$$

For which values of a and b is this a probability density function? The resulting probability distribution is called the Cauchy distribution. Does it have finite variance?

- 13.50** The exponential distribution is given by $P(x) = 1 - \exp(-\lambda x)$. Derive its probability density function, mean and the variance. This distribution often occurs in practice as a description of the time elapsing between unpredictable events.

- 13.51** The ski-lift system of Problem 5.65 is forced by wind on the mass. The force has white-noise characteristics, and $\sigma = 0.5$. Determine the probability that the maximum peak response will exceed 10° .