Mechanical Vibrations
Forced Vibration of a Single Degree of Freedom System
Harmonically Excited Vibration

- **Physical system**

  \[ m \ddot{x} + c \dot{x} + kx = F(t) \]  
  
  Equation of motion

  \( F(t) \) is a harmonic force. It may take one of the following forms:

  \[ F(t) = F_o \sin \omega t \]

  \[ F(t) = F_o \cos \omega t \]

  \[ F(t) = F_o e^{j\omega t} \]

  \( F_o \) is the amplitude of force

  \( \omega \) is the frequency of force
Harmonically Excited Vibration

Consider the form:

\[
m \ddot{x} + c \dot{x} + kx = F_0 e^{j\omega t}
\]

Equation of motion (1)

This 2nd order differential equation is non-homogeneous. Its solution has two parts:

1. Complementary function (Transient response): It is the solution for the homogeneous equation (described in previous chapter). For \(\zeta < 1\)

   The solution was written as:

   \[
   x_h(t) = e^{-\zeta \omega t} \left( C_1 \cos \omega_d t + C_2 \sin \omega_d t \right)
   \]

2. Particular integral (steady state response): It is the solution for the non-homogeneous equation. Because the force (excitation) is harmonic with frequency \(\omega\) we can expect that the response (solution) is harmonic with the same frequency \(\omega\). The solution can be written as:

   \[
   x_p(t) = \bar{X} e^{j\omega t}, \quad \bar{X} \text{ is called the 'steady state response amplitude'}
   \]
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\[ x(t) = x_h(t) + x_p(t) \]
Substitute the particular integral into equation (1) and solve for $\bar{X}$:

$$\therefore \bar{X} = \frac{F_o}{(k - \omega^2 m) + j c \omega}$$

By writing $(k - \omega^2 m) + j c \omega$ as

$$\sqrt{(k - \omega^2 m)^2 + (c \omega)^2} e^{j \phi}, \text{ where } \phi = \tan^{-1} \frac{c \omega}{k - \omega^2 m} \text{ is the "phase angle" between Force and Response.}$$

$$\therefore \bar{X} = |\bar{X}| e^{-j \phi} \quad \text{where } |\bar{X}| = \frac{F_o}{\sqrt{(k - \omega^2 m)^2 + (c \omega)^2}}$$

There for the particular integral can now be written as

$$x_p = |\bar{X}| e^{j (\omega t - \phi)}, \quad |\bar{X}| \text{ denotes the "magnitude" of the steady state response amplitude.}$$
Harmonically Excited Vibration

\[ |\bar{X}| \text{ and } \varphi \text{ may be written as:} \]

\[
|\bar{X}| = \frac{\left( \frac{F_o}{k} \right)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} , \quad \text{where} \quad r = \frac{\omega}{\omega_n} \text{ is the frequency ratio.}
\]

\[
\varphi = \tan^{-1}\left( \frac{2\zeta r}{1-r^2} \right)
\]

The total solution (response) for the equation of motion (1) is:

\[ x(t) = x_h(t) + x_p(t) \]

\[ x(t) = e^{-\zeta \omega_n t} \left( C_1 \cos \omega_d t + C_2 \sin \omega_d t \right) + |\bar{X}| e^{j(\omega t - \varphi)} \]

Note that C1 and C2 are constants to be determined from knowing the initial conditions of the motion.
Harmonically Excited Vibration

NOTES:
1. For the force form \( F(t) = F_o \sin \omega t \Rightarrow x_p(t) = |X| \sin(\omega t - \varphi) \)
2. For the force form \( F(t) = F_o \cos \omega t \Rightarrow x_p(t) = |X| \cos(\omega t - \varphi) \)
3. Transient response \( (x_h(t)) \) represents a motion that decays with time and can be neglected after a certain time.
4. Steady state response \( (x_p(t)) \) is a harmonic motion with constant amplitude and frequency \( \omega \).
5. The term \( \frac{F_o}{k} \) is usually called the "static deflection \( (\Delta_{st}) \). The ratio \( \frac{|X|}{\Delta_{st}} \) is usually called the "Magnification Factor \( (M) \)."

\[
M = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}
\]
Damped Forced Vibration System

- Graphical representation for Magnification factor $M$ and $\phi$. 

![Graphical representation for Magnification factor $M$ and $\phi$.](image)
Notes on the graphical representation of $X$.

- For $\zeta = 0$, the system is reduced and becomes *un-damped*.
- For any amount of $\zeta > 0$, the amplitude of vibration decreases (i.e. reduction in the magnification factor $M$). This is correct for any value of $r$.
- For the case of $r = 0$, the magnification factor equals 1.
- The amplitude of the forced vibration approaches zero when the frequency ratio ‘$r$’ approaches the infinity (i.e. $M \to 0$ when $r \to \infty$).
• Notes on the graphical representation for $\phi$.

- For $\zeta = 0$, the phase angle is zero for $0 < r < 1$ and $180^\circ$ for $r > 1$.
- For any amount of $\zeta > 0$ and $0 < r < 1$, $0^\circ < \phi < 90^\circ$.
- For $\zeta > 0$ and $r > 1$, $90^\circ < \phi < 180^\circ$.
- For $\zeta > 0$ and $r = 1$, $\phi = 90^\circ$.
- For $\zeta > 0$ and $r >> 1$, $\phi$ approaches $180^\circ$. 
Variation of displacement with frequency:

We have:

\[ |X| = \frac{F_o}{k} \sqrt{\left(1 - r^2\right)^2 + (2\zeta r)^2} \]

is the s.s response amplitude

1. At low frequency \((r \rightarrow 0)\) then \(|X| \approx \frac{F_o}{k}\) which means the displacement is **Stiffness control.**

2. At high frequency \((r \gg 1)\) then \(|X| \approx \frac{F_o}{m\omega^2}\) which means the displacement is **Mass control.**

3. At resonance \((r = 1)\) then \(|X| \approx \frac{F_o}{2\zeta k}\) which means the displacement is **Damping control.** (for small \(\zeta\))
Harmonically Excited Vibration

**Frequency of Maximum amplitude:**
The s.s response amplitude can be written as:

\[
|X| = \frac{F_o}{k} \left( (1 - r^2)^2 + (2\zeta r)^2 \right)^{1/2}
\]

|X| is maximum when \( \frac{d}{dr} (|X|) = 0 \)

This condition gives:

|X| is maximum when \( r = \frac{\omega}{\omega_n} = \sqrt{1 - 2\zeta^2} \) which is known as a

**Resonance Frequency,** i.e. \( \omega_{res} = \omega_n \sqrt{1 - 2\zeta^2} \)
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

\[ m\ddot{x} + kx = F_0 \cos \omega t \]

\[ x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \]

\[ x_p(t) = X \cos \omega t \]

\[ X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \]

\[ x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \]
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

\[ x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \]

Using the initial conditions \( x(t = 0) = x_0 \) and \( \dot{x}(t = 0) = \dot{x}_0 \)

\[ C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \quad C_2 = \frac{\dot{x}_0}{\omega_n} \]

\[ x(t) = \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left( \frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left( \frac{F_0}{k - m\omega^2} \right) \cos \omega t \]
Response of an Undamped System Under Harmonic Force

\[ x(t) = \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_0 t + \left( \frac{\dot{x}_0}{\omega_0} \right) \sin \omega_0 t + \left( \frac{F_0}{k - m\omega^2} \right) \cos \omega t \]

\[
X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left( \frac{\omega}{\omega_n} \right)^2}
\]

\[
\frac{X}{\delta_{st}} = \frac{1}{1 - \left( \frac{\omega}{\omega_n} \right)^2}
\]
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

\[ \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \]
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

\[ x_p(t) = \frac{F_0}{k - m\omega^2} \cos \omega t \]

- \( F(t) = F_0 \cos \omega t \)
- \( x_p(t) = X \cos \omega t \)

\[ 0 < \frac{\omega}{\omega_n} < 1. \]

- \( \frac{\omega}{\omega_n} > 1. \)
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

\[ x(t) = \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left( \frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left( \frac{F_0}{k - m\omega^2} \right) \cos \omega t \]

\[ x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[ \frac{\cos \omega t - \cos \omega_n t}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right] \]

When \( \omega / \omega_n = 1 \)

\[ \lim_{\omega \to \omega_n} \left[ \frac{\cos \omega t - \cos \omega_n t}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right] = \lim_{\omega \to \omega_n} \left[ \frac{d}{d\omega} \left( \cos \omega t - \cos \omega_n t \right) \right] = \lim_{\omega \to \omega_n} \left[ \frac{t \sin \omega t}{2 \frac{\omega}{\omega_n}^2} \right] = \frac{\omega_n t}{2} \sin \omega_n t \]
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

\[ x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \]

\[ \tau = \frac{2\pi}{\omega_n} \]
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

**Beating Phenomenon**

If the forcing frequency is close to, but not exactly equal to, the natural frequency

\[ x(t) = \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left( \frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left( \frac{F_0}{k - m\omega^2} \right) \cos \omega t \]

\[ x_0 = \dot{x}_0 = 0 \]

\[ x(t) = \frac{F_0/m}{\omega_n^2 - \omega^2} (\cos \omega t - \cos \omega_n t) = \frac{F_0/m}{\omega_n^2 - \omega^2} \left[ 2 \sin \frac{\omega + \omega_n}{2} t \cdot \sin \frac{\omega_n - \omega}{2} t \right] \]
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

Beating Phenomenon

Let the forcing frequency $\omega$ be slightly less than the natural frequency:

$$\omega_n - \omega = 2\varepsilon$$

where $\varepsilon$ is a small positive quantity. Then $\omega_n \approx \omega$ and

$$\omega + \omega_n \approx 2\omega$$

Multiplication of Eqs. (3.19) and (3.20) gives

$$\omega_n^2 - \omega^2 = 4\varepsilon \omega$$

The use of Eqs. (3.19) to (3.21) in Eq. (3.18) gives

$$x(t) = \left(\frac{F_0/m}{2\varepsilon \omega} \sin \varepsilon t\right) \sin \omega t$$
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

Beating Phenomenon

\[ x(t) \]

\[ \frac{F_0}{m} \frac{\sin \omega t}{2\epsilon \omega} \]

\[ \frac{2\pi}{\omega} \]

\[ \frac{2\pi}{\epsilon} \]
Transmissibility of displacement (support motion)

Physical system:

Mathematical model: \[ m \ddot{x} + c \left( \dot{x} - \dot{y} \right) + k(x - y) = 0 \]
Transmissibility of displacement (support motion)

Substitute the forcing function into the math. Model:

\[ m \ddot{x} + c \dot{x} + kx = kY \sin(\omega t) + c \omega Y \cos(\omega t) = A \sin(\omega t - \alpha) \]

Where:

\[ A = Y \sqrt{k^2 + (c \omega)^2}, \quad \alpha = \tan^{-1}\left(-\frac{c \omega}{k}\right) \]

Assume \( x_p(t) = X \sin(\omega t - \alpha - \phi_1) \) (Harmonic motion)

\[ x_p(t) = \frac{Y \sqrt{k^2 + (c \omega)^2}}{\left[(k - m\omega^2)^2 + (c \omega)^2\right]^{1/2}} \sin(\omega t - \phi_1 - \alpha) \]

\[ \phi_1 = \tan^{-1}\left(\frac{c \omega}{k - m\omega^2}\right) \]
Transmissibility of displacement (support motion)

\[ x_p(t) = X \sin(\omega t - \phi) \]

\[
\frac{X}{Y} = \left[ \frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2} \right]^{1/2} = \left[ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}
\]

\[
\phi = \tan^{-1} \left[ \frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2} \right] = \tan^{-1} \left[ \frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2} \right]
\]

The ratio of the amplitude of the response \( x_p(t) \) to that of the base motion \( y(t) \), \( \frac{X}{Y} \), is called the displacement transmissibility. \( \frac{X}{Y} \equiv T_d \)
Transmissibility of displacement (support motion)

Graphical representation of Force or Displacement Transmissibility ((TR) and the Phase angle (\(\phi\))
The following aspects of displacement transmissibility, $T_d = \frac{X}{Y}$:

1. The value of $T_d$ is unity at $r = 0$ and close to unity for small values of $r$.
2. For an undamped system ($\zeta = 0$), $T_d \to \infty$ at resonance ($r = 1$).
3. The value of $T_d$ is less than unity ($T_d < 1$) for values of $r > \sqrt{2}$ (for any amount of damping $\zeta$).
4. The value of $T_d$ is unity for all values of $\zeta$ at $r = \sqrt{2}$.
5. For $r < \sqrt{2}$, smaller damping ratios lead to larger values of $T_d$. On the other hand, for $r > \sqrt{2}$, smaller values of damping ratio lead to smaller values of $T_d$.
6. The displacement transmissibility, $T_d$, attains a maximum for $0 < \zeta < 1$ at the frequency ratio $r = r_m < 1$ given by (see Problem 3.60):

$$r_m = \frac{1}{2\zeta} \left[ \sqrt{1 + 8\zeta^2} - 1 \right]^{1/2}$$
Transmissibility of displacement (support motion)

- **Force Transmitted**

\[
F = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x}
\]

\[
x_p(t) = X \sin(\omega t - \phi)
\]

\[
F = m\omega^2 X \sin(\omega t - \phi) = F_T \sin(\omega t - \phi)
\]
Transmissibility of displacement (support motion)

\[ X = Y \left[ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} \]

\[ \frac{F_T}{kY} = r^2 \left[ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} \]
Transmissibility of displacement (support motion)

Relative Motion

\[ m \ddot{x} + c (\dot{x} - \dot{y}) + k (x - y) = 0 \]

\[ z = x - y \rightarrow m \ddot{z} + c \dot{z} + kz = -m \ddot{y} = m \omega^2 Y \sin \omega t \]

\[ z(t) = \frac{m \omega^2 Y \sin(\omega t - \phi_1)}{\left[ (k - m \omega^2)^2 + (c \omega)^2 \right]^{1/2}} = Z \sin(\omega t - \phi_1) \]

\[ Z = \frac{m \omega^2 Y}{\sqrt{(k - m \omega^2)^2 + (c \omega)^2}} = Y \frac{r^2}{\sqrt{(1 - r^2)^2 + (2 \zeta r)^2}} \]

\[ \phi_1 = \tan^{-1} \left( \frac{c \omega}{k - m \omega^2} \right) = \tan^{-1} \left( \frac{2 \zeta r}{1 - r^2} \right) \]
Transmissibility of displacement (support motion)
Forced Vibration due to Rotating Unbalance

Unbalance in rotating machines is a common source of vibration excitation. If $M_t$ is the total mass of the system, $m$ is the eccentric mass and $\omega$ is the speed of rotation, the centrifugal force due to unbalanced mass is $me\omega^2$ where $e$ is the eccentricity.

- The vertical component ($me\omega^2 \sin(\omega t)$) is the effective one because it is in the direction of motion of the system. The equation of motion is:

$$M_t \ddot{x} + c \dot{x} + kx = me\omega^2 \sin(\omega t)$$
Forced Vibration due to Rotating Unbalance

\[
|\bar{X}| = \frac{\left( \frac{me \omega^2}{k} \right)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \quad \text{or}
\]

\[
|\bar{X}| = \frac{\left( \frac{me}{M_t} \right) r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}
\]

and

\[
\varphi = \tan^{-1}\left( \frac{2\zeta r}{1-r^2} \right)
\]
Forced Vibration due to Rotating Unbalance

\[
\frac{M X}{m e} \text{ (Rotating unbalance)} = f(r, \zeta)
\]

\[ r = \frac{\omega}{\omega_n} \]

Where:
- \( r \) is the reduced frequency
- \( \omega \) is the angular frequency
- \( \omega_n \) is the natural frequency
- \( \zeta \) is the damping ratio
Forced Vibration due to Rotating Unbalance

1. All the curves begin at zero amplitude. The amplitude near resonance \( \omega = \omega_n \) is markedly affected by damping. Thus if the machine is to be run near resonance, damping should be introduced purposefully to avoid dangerous amplitudes.

2. At very high speeds \( (\omega \text{ large}) \), \( \frac{MX}{me} \) is almost unity, and the effect of damping is negligible.

3. For \( 0 < \zeta < \frac{1}{\sqrt{2}} \), the maximum of \( \frac{MX}{me} \) occurs when

\[
\frac{d}{dr} \left( \frac{MX}{me} \right) = 0
\]

The solution of Eq. (3.82) gives

\[
r = \frac{1}{\sqrt{1 - 2\zeta^2}} > 1
\]

with the corresponding maximum value of \( \frac{MX}{me} \) given by

\[
\left( \frac{MX}{me} \right)_{\text{max}} = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}
\]

Thus the peaks occur to the right of the resonance value of \( r = 1 \).

4. For \( \zeta > \frac{1}{\sqrt{2}} \), \( \left[ \frac{MX}{me} \right] \) does not attain a maximum. Its value grows from 0 at \( r = 0 \) to 1 at \( r \to \infty \).
If the deflection of foundation is negligible, then the force transmitted to the foundation is
\[ F_{tr} e^{j\omega t} = kx + cx \]
Assume harmonic motion i.e., \[ x = X e^{j\omega t} \]

\[ F_{tr} = \frac{(k + j\omega c)me\omega^2}{((k - \omega^2M_t) + j\omega c)} \]

\[ |F_{tr}| = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} me\omega^2 \]

\[ \rightarrow TR_F = \frac{|F_{tr}|}{me\omega_n^2} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} r^2 \]
Transmissibility of Force
Example 3.1: Plate Supporting a Pump:

A reciprocating pump, weighing 68 kg, is mounted at the middle of a steel plate of thickness 1 cm, width 50 cm, and length 250 cm. clamped along two edges as shown in Fig. During operation of the pump, the plate is subjected to a harmonic force, \( F(t) = 220 \cos (62.832t) \) N. if \( E=200 \) Gpa, \textit{Find the amplitude of vibration of the plate.}
• Example 3.1: solution

• The plate can be modeled as fixed – fixed beam has the following stiffness:

\[ k = \frac{192EI}{l^3} \]

But \( I = \frac{1}{12}bh^3 = \frac{1}{12}(50 \times 10^{-2})(1 \times 10^{-2})^3 = 41.667 \times 10^{-9} m^4 \)

So, \( k = \frac{192(200 \times 10^9)(41.667 \times 10^{-9})}{(250 \times 10^{-2})^3} = 102,400.82 \text{ N/m} \)

• The maximum amplitude (X) is found as:

\[ X = \frac{F_o}{k - m\omega^2} = \frac{220}{102,400.82 - 68(62.832)} = -1.32487 \text{ mm} \]
Example 3.2:

Find the total response of a single-degree-of-freedom system with \( m = 10 \) kg, \( c = 20 \) N-s/m, \( k = 4000 \) N/m, \( x_0 = 0.01 \) m and \( \dot{x} = 0 \) when an external force \( F(t) = F_0 \cos(\omega t) \) acts on the system with \( F_0 = 100 \) N and \( \omega = 10 \) rad/sec.

Solution

a. From the given data

\[
\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}
\]

\[
\zeta = \frac{c}{2m\omega_n} = \frac{20}{2(10)(20)} = 0.05
\]

\[
\omega_d = \sqrt{1 - \zeta^2} \omega_n
\]

\[
\omega_d = \sqrt{1 - 0.05^2} (20) = 19.975 \text{ rad/s}
\]

\[
r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5
\]
Example 3.2:
Solution

\[ \delta_{st} = \frac{F_o}{k} = \frac{100}{4000} = 0.025m \]

\[ |X| = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 0.3326m \]

\[ \phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = 3.814^\circ \]

• Total solution:
• \( X(t) = X_h(t) + X_p(t) \)

\[ x(t) = Ae^{-\zeta \omega_n t} \cos(\omega_d t - \psi) + |X| \cos(\omega t - 0.066) \]

\[ x(t) = Ae^{-0.05*20t} \cos(19.97t - \psi) + 0.3326\cos(20t - 0.066) \]
at \( t = 0 \), \( x(t) = 0.01m \)

\[ 0 = A\cos \psi + 0.3326 \cos 0.066 \]

\( \therefore A\cos \psi = -0.33187 \quad \text{(1)} \)

\[ x(t) = Ae^{-t} \cos(19.97t - \psi) + 0.3326 \cos(20t - 0.066) \]

\[ \dot{x}(t) = Ae^{-t} (-19.97 \sin (19.97t - \psi)) + A \cos (19.97t - \psi)(-t)e^{-t} \]

\[-0.3326 \times 20 \sin (20t - 0.066) \]

at \( t = 0 \), \( \dot{x}(t) = 0 \)

\( \therefore 0 = 19.97A \sin \psi + 0.438, \quad A \sin \psi = -0.0219 \quad \text{(2)} \)

From (1) and (2) \( \psi = 0.066 \text{ rad} \) and \( A = -0.3325 \)

\( \therefore x(t) = -6.64e^{-t} \cos(19.97t - 0.066) - 0.3326 \cos(20t - 0.066) \)
**Viscous Damping**

\[ F(t) = F_o \sin \omega t \]

(b) Free-body diagram

\[ F = kx + cx \]
Viscous Damping

\[ x = X \sin (\omega t - \phi) \]

\[ \dot{x} = \omega X \cos (\omega t - \phi) = \pm \omega X \sqrt{1 - \sin^2 (\omega t - \phi)} \]

\[ = \pm \omega \sqrt{X^2 - x^2} \]

\[ F_d = c \dot{x} = \pm c \omega \sqrt{X^2 - x^2} \]

\[ \left( \frac{F_d}{c \omega X} \right)^2 + \left( \frac{x}{X} \right)^2 = 1 \]
Viscous Damping

\[ F = kx + cx \dot{x} \]

\[ x = X \sin(\omega t - \phi) \]

\[ F = kX \sin(\omega t - \phi) + cX \omega \cos(\omega t - \phi) \]

\[ = kx \pm c \omega \sqrt{X^2 - (X \sin(\omega t - \phi))^2} \]

\[ = kx \pm c \omega \sqrt{X^2 - x^2} \]
Viscous Damping

\[ W_d = \int c\dot{x} \, dx = \int c\dot{x}^2 \, dt \]

\[ = c\omega^2X^2 \int_0^{2\pi/\omega} \cos^2(\omega t - \phi) \, dt = \pi c\omega X^2 \]
Energy Dissipated by Damping

Hysteresis damping

\[ \Delta W = \pi h X^2 \]
EQUIVALENT VISCOUS DAMPING

hysteresis damping

\[ F(t) = F_0 \sin \omega t \]
**EQUIVALENT VISCOUS DAMPING**

\[ W_{d-v} = W_{d-s} \]

\[ \pi C_{eq} \omega X^2 = \pi hX^2 \] \[ \Rightarrow \quad C_{eq} = \frac{h}{\omega} \]

\[ F = kx + cx' \]

\[ x = X e^{i\omega t} \]

\[ F = kX e^{i\omega t} + c\omega iX e^{i\omega t} = (k + i\omega c) x \]

\[ F = (k + ih)x \]

\[ k + ih = k \left(1 + i\frac{h}{k}\right) = k(1 + i\beta) \] \[ \Rightarrow \quad h = \beta k \]
**EQUIVALENT VISCOUS DAMPING**

\[ W_{d-v} = W_{d-s} \]

\[ \pi C_{eq} \omega X^2 = \pi h X^2 \]

\[ C_{eq} = \frac{h}{\omega} \]

\[ \left( \frac{h}{\omega} \right) \ddot{x} - kx + F_0 \sin(\omega t) = m\ddot{x} \]

\[ m\ddot{x} + \left( \frac{h}{\omega} \right) \dot{x} + kx = F_0 \sin(\omega t) \]
EQUIVALENT VISCOUS DAMPING

\[ m \ddot{x} + \left( \frac{h}{\omega} \right) \dot{x} + kx = F_0 \sin(\omega t) \]

\[ m \ddot{x} + \frac{\beta k}{\omega} \dot{x} + kx = F_0 \sin \omega t \]

\[ x_p(t) = X \sin (\omega t - \phi) \]

\[ X = \frac{F_0}{\left[ (k - m\omega^2)^2 + c^2\omega^2 \right]^{1/2}} \]

\[ \phi = \tan^{-1}\left( \frac{c\omega}{k - m\omega^2} \right) \]
EQUIVALENT VISCOUS DAMPING

\[ m\ddot{x} + \frac{\beta k}{\omega} \dot{x} + kx = F_0 \sin \omega t \]

\[ x_p(t) = X \sin(\omega t - \phi) \]

\[ X = \frac{F_0}{k \left[ \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \beta^2 \right]^{1/2}} \]

\[ \phi = \tan^{-1} \left[ \frac{\beta}{1 - \frac{\omega^2}{\omega_n^2}} \right] \]
EQUIVALENT VISCOUS DAMPING

\[ X = \frac{F_0}{k \left[ \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \beta^2 \right]^{1/2}} \]

\[ \phi = \tan^{-1} \left[ \frac{\beta}{\left( 1 - \frac{\omega^2}{\omega_n^2} \right)} \right] \]

Graphs showing the relationship between \( \frac{X}{(F_0/k)} \) and \( \frac{\omega}{\omega_n} \) for different values of \( \beta \):
- \( \beta = 0 \)
- \( \beta = 0.2 \)
- \( \beta = 0.5 \)
- \( \beta = 1.0 \)
EQUIVALENT VISCIOUS DAMPING

Coulomb damping

\[ \Delta E = 4R x_0 \]
Coulomb damping

\[ F(t) = F_0 \sin \omega t \]

\[ m\ddot{x} + kx \pm \mu N = F(t) = F_0 \sin \omega t \]
EQUIVALENT VISCOUS DAMPING

\[ m\ddot{x} + kx \pm \mu N = F(t) = F_0 \sin \omega t \]

\[ \Delta W = 4\mu NX \]

\[ \Delta W = \pi c_{eq}\omega X^2 \]

\[ c_{eq} = \frac{4\mu N}{\pi \omega X} \]

\[ m\ddot{x} + \left( \frac{4\mu N}{\pi \omega X} \right) \dot{x} + kx = F_0 \sin(\omega t) \]

\[ x_p(t) = X \sin(\omega t - \phi) \]

\[ X = \left[ \frac{F_0/k}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{4\mu N}{\pi kX}\right)^2} \right]^{1/2} \]

\[ \phi = \tan^{-1} \left( \frac{c_{eq}\omega}{k - m\omega^2} \right) = \tan^{-1} \left( \frac{4\mu N}{\pi kX} \right) \left( \frac{1 - \frac{\omega^2}{\omega_n^2}}{\omega_n^2} \right) \]
EQUIVALENT VISCOUS DAMPING

\[ X = \frac{F_0}{k} \left[ 1 - \left( \frac{4\mu N}{\pi F_0} \right)^2 \right]^{1/2} \]

\[ \phi = \tan^{-1} \left[ \frac{4\mu N}{\pi F_0} \right]^{1/2} \left( 1 - \left( \frac{4\mu N}{\pi F_0} \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 \right]^{1/2} \]