

- 2.1. An industrial press is mounted on a rubber pad to isolate it from its foundation. If the rubber pad is compressed 5 mm by the self-weight of the press, find the natural frequency of the system.

$$\omega_n = \sqrt{\frac{k}{m}} \quad , \quad k\delta_{st} = mg \quad \rightarrow \quad \frac{k}{m} = \frac{g}{\delta_{st}}$$

$$\rightarrow \omega_n = \sqrt{\frac{g}{\delta_{st}}}$$

$$\delta_{st} = 5 \times 10^{-3} \text{ m}$$

$$\omega_n = \left(\frac{g}{\delta_{st}} \right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}} \right)^{1/2} = 44.2945 \text{ rad/sec} = 7.0497 \text{ Hz}$$

- 2.2. A spring-mass system has a natural period of 0.21 sec. What will be the new period if the spring constant is (i) increased by 50% and (ii) decreased by 50%?

$$\tau_n = 0.21 \text{ sec} = 2\pi \sqrt{\frac{m}{k}} \quad , \quad \sqrt{m} = \frac{0.21 \sqrt{k}}{2\pi}$$

$$(i) (\tau_n)_{\text{new}} = \frac{2\pi \sqrt{m}}{\sqrt{k_{\text{new}}}} = \frac{2\pi \sqrt{m}}{\sqrt{1.5k}} = \frac{2\pi \left(\frac{0.21 \sqrt{k}}{2\pi} \right)}{\sqrt{1.5k}} = 0.1715 \text{ sec.}$$

$$(ii) (\tau_n)_{\text{new}} = \frac{2\pi \sqrt{m}}{\sqrt{k_{\text{new}}}} = \frac{2\pi \sqrt{m}}{\sqrt{0.5k}} = 2\pi \left(\frac{0.21 \sqrt{k}}{2\pi} \right) \frac{1}{\sqrt{0.5k}} = 0.2970 \text{ sec.}$$

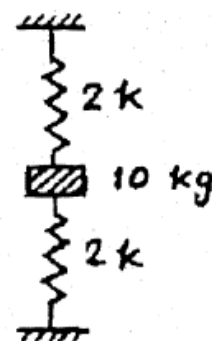
- 2.4. A helical spring, when fixed at one end and loaded at the other, requires a force of 100 N to produce an elongation of 10 mm. The ends of the spring are now rigidly fixed, one end vertically above the other, and a mass of 10 kg is attached at the middle point of its length. Determine the time taken to complete one vibration cycle when the mass is set vibrating in the vertical direction.

$$k = 100 / \left(\frac{10}{1000} \right) = 10000 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4k}{m}} = \left(\frac{4 \times 10^4}{10} \right)^{1/2}$$

$$= 63.2456 \text{ rad/sec}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993 \text{ sec}$$



2.5. The maximum velocity attained by the mass of a simple harmonic oscillator is 10 cm/sec, and the period of oscillation is 2 sec. If the mass is released with an initial displacement of 2 cm, find (a) the amplitude, (b) the initial velocity, (c) the maximum acceleration, and (d) the phase angle.

$$x = A \cos(\omega_n t - \phi_0) \quad , \quad \dot{x} = -\omega_n A \sin(\omega_n t - \phi_0) \quad ,$$

$$\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_0)$$

$$(a) \quad \omega_n A = 0.1 \text{ m/sec} \quad ; \quad \tau_n = \frac{2\pi}{\omega_n} = 2 \text{ sec}, \quad \omega_n = 3.1416 \text{ rad/sec}$$

$$A = 0.1 / \omega_n = 0.03183 \text{ m}$$

$$(d) \quad x_0 = x(t=0) = A \cos(-\phi_0) = 0.02 \text{ m}$$

$$\cos(-\phi_0) = \frac{0.02}{A} = 0.6283$$

$$\phi_0 = 51.0724^\circ$$

$$(b) \quad \dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1 \sin(-51.0724^\circ) \\ = 0.07779 \text{ m/sec}$$

$$(c) \quad \ddot{x}|_{\max} = \omega_n^2 A = (3.1416)^2 (0.03183) = 0.314151 \text{ m/sec}^2$$

- 2.6. Three springs and a mass are attached to a rigid, weightless, bar PQ as shown in Fig. 2.29. Find the natural frequency of vibration of the system.

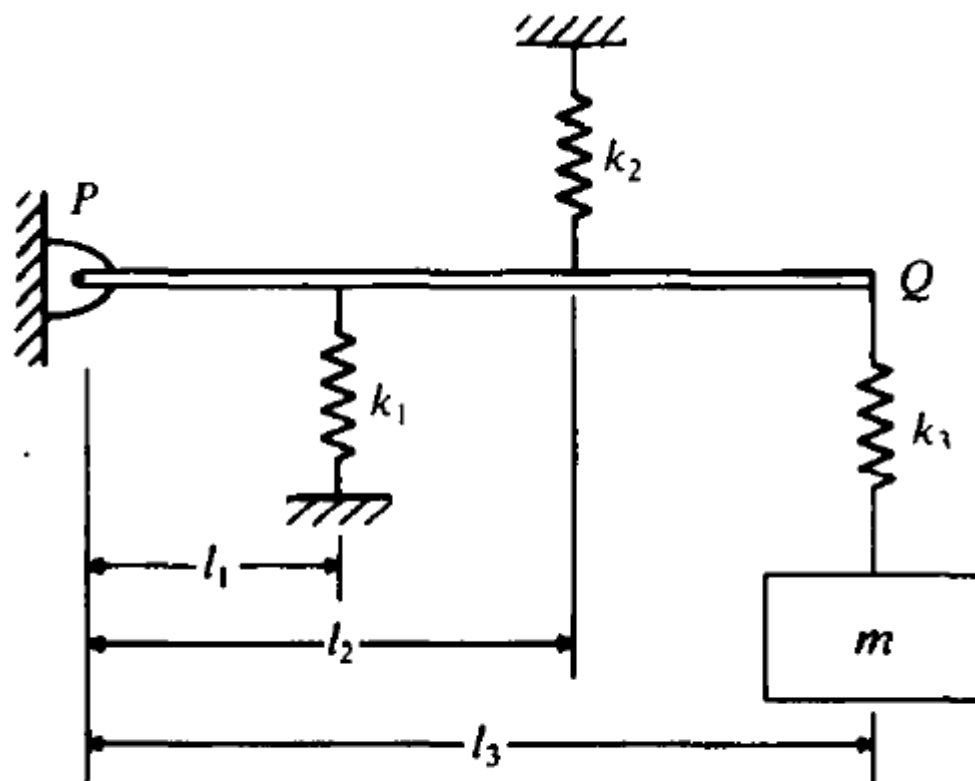


Figure 2.29

For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2$$

$$\text{i.e., } (k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2) / l_3^2$$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

$$k_{eq} = \frac{(k_{12})_{eq} k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{ k_1 \left(\frac{l_1}{l_3} \right)^2 + k_2 \left(\frac{l_2}{l_3} \right)^2 \right\} k_3}{k_1 \left(\frac{l_1}{l_3} \right)^2 + k_2 \left(\frac{l_2}{l_3} \right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m (k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

- 2.8. Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane, as shown in Fig. 2.30.

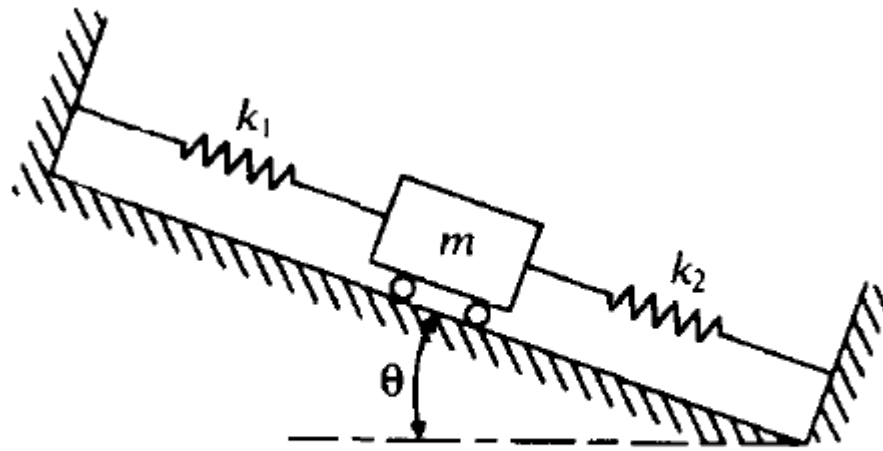


Figure 2.30

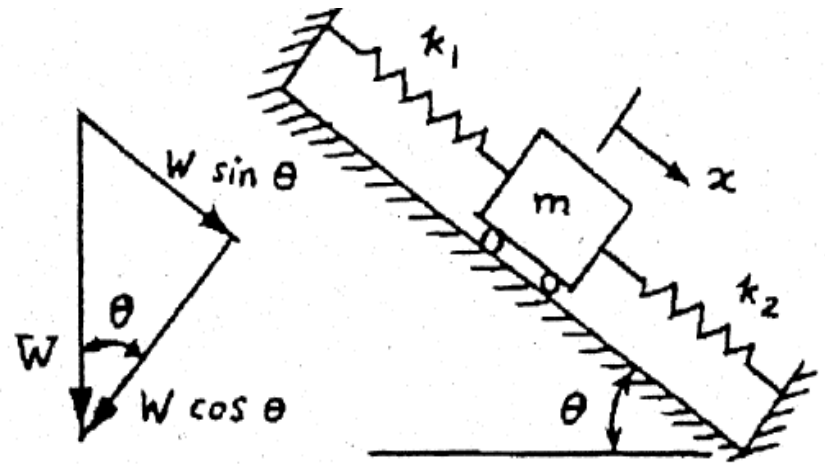
Let x be measured from the position of mass at which the springs are unstretched.

Equation of motion is

$$m\ddot{x} = -k_1(x + \delta_{st}) - k_2(x + \delta_{st}) + W \sin \theta \quad \text{--- (E}_1\text{)}$$

where $\delta_{st} (k_1 + k_2) = W \sin \theta$.

Thus Eq. (E₁) becomes $m\ddot{x} + (k_1 + k_2)x = 0 \Rightarrow \omega_n = \sqrt{\frac{k_1 + k_2}{m}}$.



- 2.11. A rigid block of mass M is mounted on four elastic supports as shown in Fig. 2.33. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each elastic support is k , find the natural frequency of vibration of the system (a) without the mass m , and (b) with the mass m . Also find the resulting motion of the system in case (b).

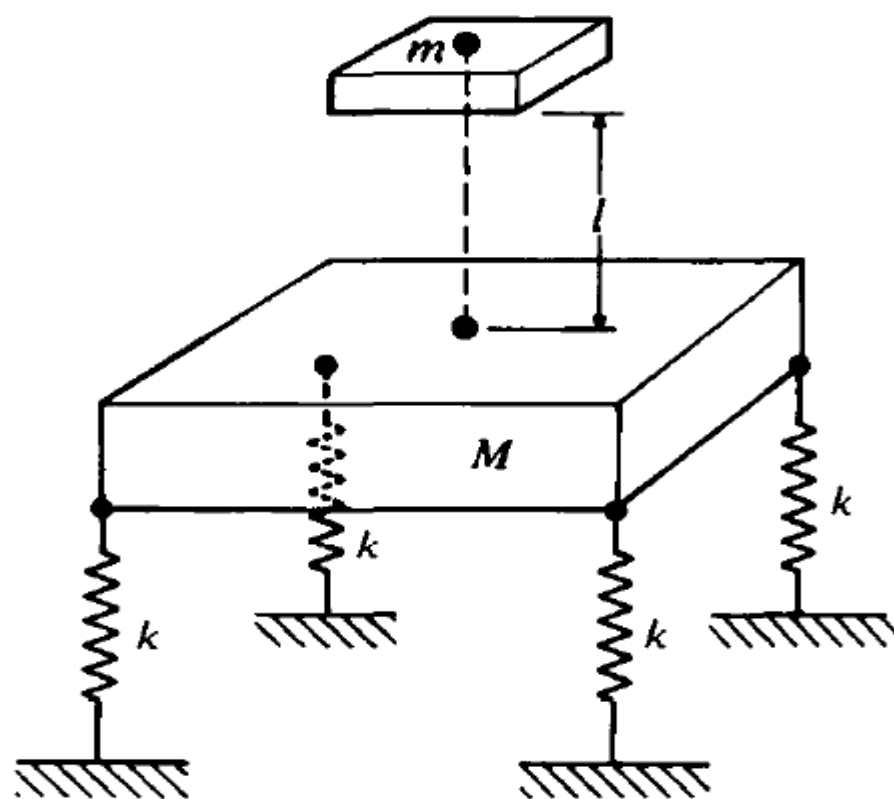


Figure 2.33

$$(a) \quad \omega_n = \sqrt{4k/M}$$

$$(b) \quad \omega_n = \sqrt{4k/(M+m)}$$

Initial conditions:

velocity of falling mass $m = v = \sqrt{2gl}$ ($\because v^2 - u^2 = 2gl$)
 $x=0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum: $(M+m) \dot{x}_0 = m v = m \sqrt{2gl}$
 $\dot{x}_0 = \dot{x}(t=0) = \frac{m}{M+m} \sqrt{2gl}$

Complete solution: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where $A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16 k^2} + \frac{m^2 g l}{2k(M+m)}}$

and $\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-\sqrt{g}}{\sqrt{8l k (M+m)}}$

- 2.14. The natural frequency of a spring-mass system is found to be 2 Hz. When an additional mass of 1 kg is added to the original mass m , the natural frequency is reduced to 1 Hz. Find the spring constant k and the mass m .

$$\omega_n = 2 \text{ Hz} = 12.5664 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$\sqrt{k} = 12.5664 \sqrt{m}$$

$$\omega_n' = \sqrt{\frac{k'}{m'}} = \sqrt{\frac{k}{m+1}} = 6.2832 \text{ rad/sec}$$

$$\sqrt{k} = 6.2832 \sqrt{m+1}$$

$$= 12.5664 \sqrt{m}$$

$$\sqrt{m+1} = 2 \sqrt{m}, \quad m = \frac{1}{3} \text{ kg}$$

$$k = (12.5664)^2 m = 52.6381 \text{ N/m}$$

- 2.23. A helical spring of stiffness k is cut into two halves and a mass m is connected to the two halves as shown in Fig. 2.43(a). The natural time period of this system is found to be 0.5 sec. If an identical spring is cut so that one part is $\frac{1}{4}$ and the other part $\frac{3}{4}$ of the original length, and the mass m is connected to the two parts as shown in Fig. 2.43(b), what would be the natural period of the system?

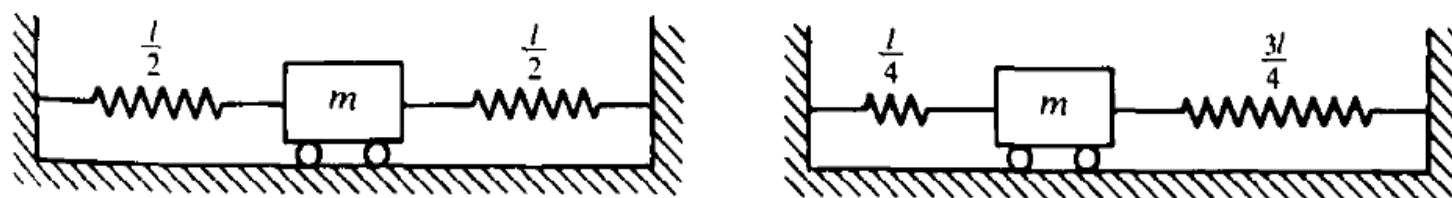
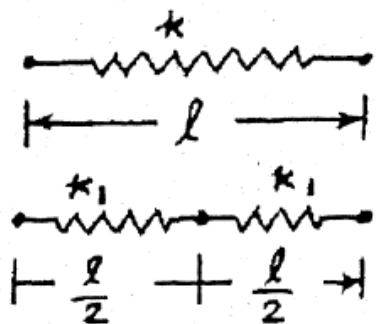


Figure 2.43



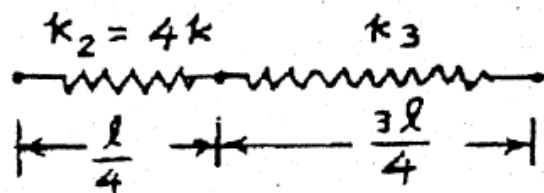
$$\frac{1}{k_{\text{total}}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{\text{total}} = \frac{k_1}{2} \equiv k; \quad k_1 = 2k$$

$$T_n = 2\pi \sqrt{\frac{m}{k_{\text{eq}}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{\text{total}}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

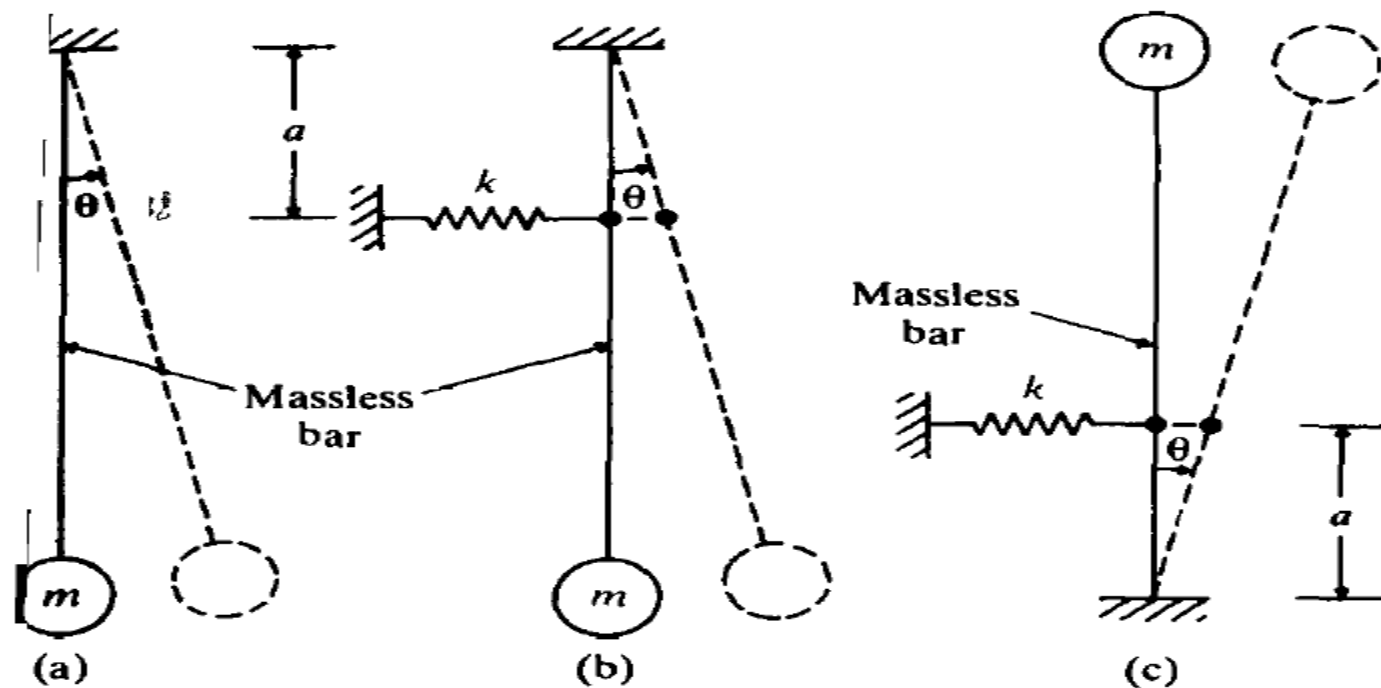
$$k_3 = \frac{4}{3}k$$

$$T_n = 2\pi \sqrt{\frac{m}{k_{\text{eq}}}}$$

$$\text{where } k_{\text{eq}} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

$$\therefore T_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4} \sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4} \left(\frac{1}{2\pi} \right) = 0.4330 \text{ sec}$$

A mass m is attached at the end of a bar of negligible mass and is made to vibrate in three different configurations, as indicated in Figs. 2.47(a) to (c). Find the configuration corresponding to the highest natural frequency.



2.47

$$(a) \quad \omega_n = \sqrt{\frac{g}{l}}$$

$$(b) \quad ml^2 \ddot{\theta} + ka^2 \sin \theta + mgl \sin \theta = 0 ; \quad ml^2 \ddot{\theta} + (ka^2 + mgl) \theta = 0$$

$$\omega_n = \sqrt{\frac{ka^2 + mgl}{ml^2}}$$

$$(c) \quad ml^2 \ddot{\theta} + ka^2 \sin \theta - mgl \sin \theta = 0 ; \quad ml^2 \ddot{\theta} + (ka^2 - mgl) \theta = 0$$

$$\omega_n = \sqrt{\frac{ka^2 - mgl}{ml^2}}$$

configuration (b) has the highest natural frequency.

- 2.32. A cylinder of mass m and mass moment of inertia J_0 is free to roll without slipping but is restrained by two springs of stiffnesses k_1 and k_2 as shown in Fig. 2.50. Find its natural frequency of vibration. Also find the value of a that maximizes the natural frequency of vibration.

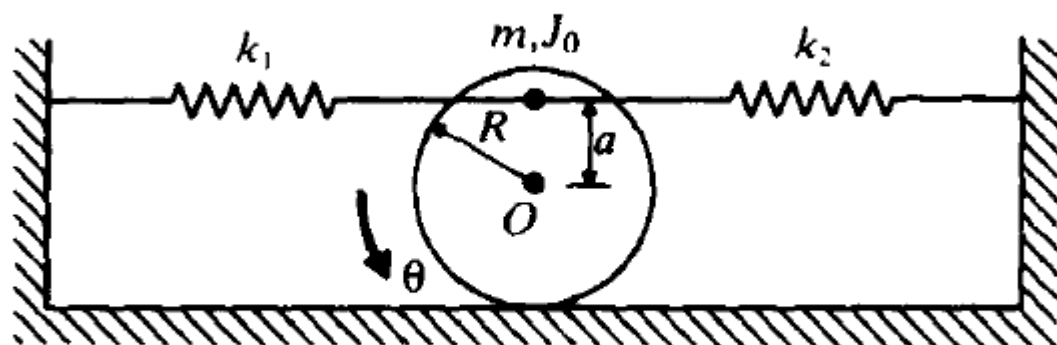


Figure 2.50

$$J_O = \frac{1}{2} m R^2, \quad J_C = \frac{1}{2} m R^2 + m R^2$$

Let angular displacement = θ

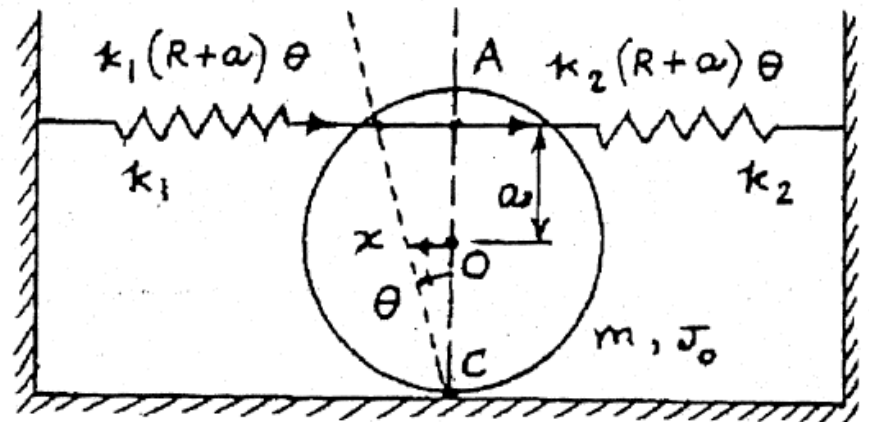
Equation of motion:

$$J_C \ddot{\theta} + k_1 (R+a)^2 \theta + k_2 (R+a)^2 \theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1 + k_2) (R+a)^2}{J_C}} = \sqrt{\frac{(k_1 + k_2) (R+a)^2}{1.5 m R^2}} \quad (E_1)$$

Equation (E₁) shows that ω_n increases with the value of a .

$\therefore \omega_n$ will be maximum when $a = R$.



- 2.49. A locomotive car of mass 2000 kg traveling at a velocity $v = 10$ m/sec is stopped at the end of tracks by a spring-damper system as shown in Fig. 2.54. If the stiffness of

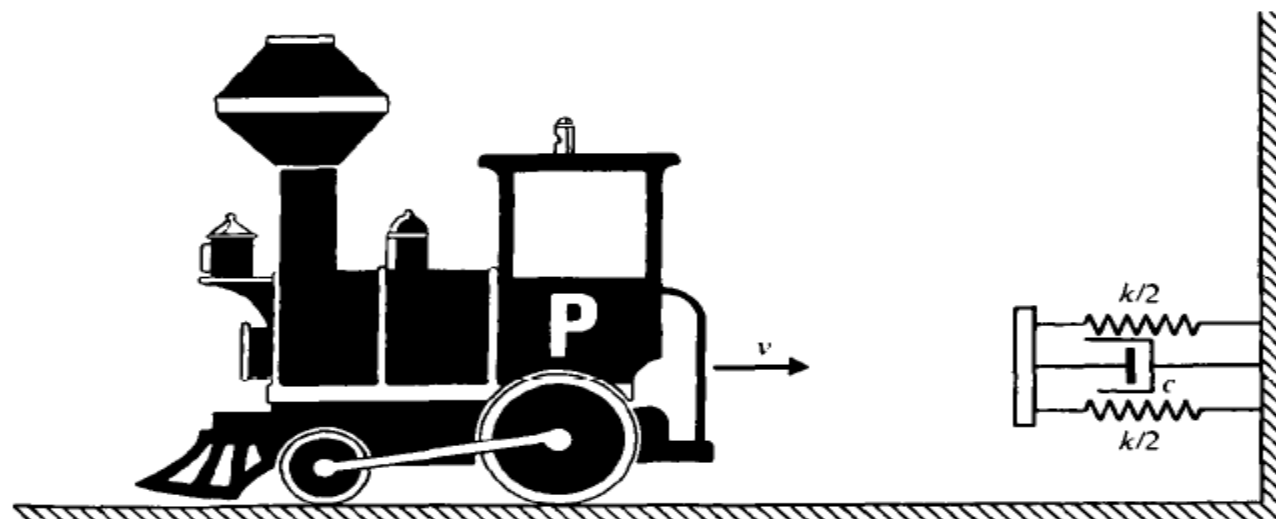


Figure 2.54

- the spring is $k = 40$ N/mm and the damping constant is $c = 20$ N-s/mm, determine
- the maximum displacement of the car after engaging the springs and damper and
 - the time taken to reach the maximum displacement.

$$m = 2000 \text{ kg}, \quad v = \dot{x}_0 = 10 \text{ m/sec}, \quad k = 40,000 \text{ N/m}$$

$$c = 20,000 \text{ N-sec/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721 \text{ rad/sec}$$

$$c_c = 2\sqrt{km} = 25,298.221 \text{ N-sec/m}$$

$$\zeta = c/c_c = 0.7906$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4.4721 \sqrt{1 - (0.7906)^2} = 2.7384 \text{ rad/sec}$$

$$\tau_d = 2\pi/\omega_d = 2.2945 \text{ sec}$$

(a) For $x_0 = 0$ and $\dot{x}_0 = 10 \text{ m/sec}$, Eq. (2.72) gives

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n \sqrt{1 - \zeta^2}} \sin \omega_n \sqrt{1 - \zeta^2} t$$

$$\text{At } x_{\max}, \quad \omega_n t \simeq \frac{\pi}{2} \text{ and } \sin \omega_n \sqrt{1 - \zeta^2} t \simeq 1$$

$$\therefore x_{\max} \simeq e^{-0.7906(\pi/2)} \cdot \left(\frac{10}{2.7384} \right) \cdot (1) = 1.0548 \text{ m}$$

$$(b) \quad t = \tau_d/4 = 2.2945/4 = 0.5736 \text{ sec.}$$

- 2.50. A torsional pendulum has a natural frequency of 200 cycles/min when vibrating in vacuum. The mass moment of inertia of the disc is 0.2 kg-m^2 . It is then immersed in oil and its natural frequency is found to be 180 cycles/min. Determine the damping constant. If the disc, when placed in oil, is given an initial displacement of 2° , find its displacement at the end of the first cycle.

$$\omega_n = 200 \text{ cycles/min} = 20.944 \text{ rad/sec}, \quad \omega_d = 180 \text{ cycles/min} = 18.8496 \frac{\text{rad}}{\text{sec}}$$

$$J_0 = 0.2 \text{ kg-m}^2$$

$$\text{Since } \omega_d = \sqrt{1-\gamma^2} \omega_n, \quad \gamma = \sqrt{1 - \left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1 - \left(\frac{18.8496}{20.944}\right)^2} = 0.4359$$

$$= \frac{c_t}{(c_t)_{\text{cri}}} = \frac{c_t}{2 J_0 \omega_n}$$

$$c_t = 2 J_0 \omega_n \gamma = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \text{ N-m-s/rad}$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_0 = 0$, $\theta_0 = 2^\circ = 0.03491 \text{ rad}$ and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333 \text{ sec}$,

$$\theta(t) = e^{-\gamma \omega_n t} \theta_0 \left\{ \cos \omega_d t + \frac{\gamma \omega_n}{\omega_d} \sin \omega_d t \right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)} (0.03491) \left\{ \cos 18.8496 \times 0.3333 \right.$$

$$\left. + \frac{0.4359 \times 20.944}{18.8496} \sin 18.8496 \times 0.3333 \right\}$$

$$= 0.001665 \text{ rad} = 0.09541^\circ$$

- 2.51. A body vibrating with viscous damping makes 5 complete oscillations per second, and in 50 cycles its amplitude diminishes to 10%. Determine the logarithmic decrement and the damping ratio. In what proportion will the period of vibration be decreased if damping is removed?

$$\tau_d = 0.2 \text{ sec} = \frac{2\pi}{\omega_d}, \quad \omega_d = 31.416 \text{ rad/sec}$$

$$\text{From Eq. (2.92)} \quad \delta = \frac{1}{50} \ln 10 = 0.04605$$

$$\gamma = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$$\omega_n = \omega_d / \sqrt{1 - \gamma^2} = 31.417 \text{ rad/sec}; \quad \tau_n = \frac{2\pi}{\omega_n} = 0.19999 \text{ sec}$$

$$\text{Proportional decrease in period} = \left(\frac{0.2 - 0.19999}{0.2} \right) = 0.00005$$

- 2.53. A viscously damped system has a stiffness of 5000 N/m, critical damping constant of 0.2 N-s/mm, and a logarithmic decrement of 2.0. If the system is given an initial velocity of 1 m/sec, determine the maximum displacement of the system.

$$k = 5000 \text{ N/m}, \quad c_c = 0.2 \text{ N-s/mm} = 200 \text{ N-s/m}$$

$$= 2 \sqrt{k m} = 2 \sqrt{5000 m}$$

$$m = 2 \text{ kg}$$

$$\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50 \text{ rad/sec}$$

$$\text{Logarithmic decrement} = \delta = \frac{2\pi\gamma}{\sqrt{1-\gamma^2}} = 2.0$$

$$\text{i.e., } \gamma = \frac{c}{c_c} = 0.3033 \quad \text{and} \quad c = 0.3033 (0.2) = 60.66 \text{ N-s/m}$$

Assuming $x_0 = 0$ and $\dot{x}_0 = 1 \text{ m/s}$,

$$x(t) = e^{-\gamma\omega_n t} \frac{\dot{x}_0}{\omega_n \sqrt{1-\gamma^2}} \sin \sqrt{1-\gamma^2} \omega_n t$$

For $x_{\max}$, $\omega_n t \approx \pi/2$ and $\sin \sqrt{1-\gamma^2} \omega_n t \approx 1$

$$\therefore x_{\max} \approx e^{-0.3033(\pi/2)} \frac{1}{50 \sqrt{1-0.3033^2}} (1) = 0.01303 \text{ m}$$